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A SERIES SOLUTION ALGORITHM FOR INITIAL BOUNDARY-VALUE PROBLEMS WITH VARIABLE PROPERTIES

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Abstract—A multiple infinite trigonometric cum polynomial series method for solving initial-boundary value problems governed by hyperbolic differential equations with variable coefficients is developed. The method proposed herein can be easily applied to a broad class of engineering systems including those cases where boundary conditions may vary with time. In the proposed mathematical technique, the solution form is assumed as a combination of infinite Fourier series and polynomial series of nth order, where n is the order of the differential equation. The coefficients of the polynomial series are obtained as functions of undetermined Fourier series coefficients by satisfying the initial-boundary conditions. The variable coefficients are expanded in appropriate half-range sine or cosine series. Insertion of the above Fourier-polynomial series solutions into the differential equation and application of orthogonality conditions leads to a linear summation equation which can be solved in open form. However, the authors have developed a closed-form series solution consisting of a highly efficient algorithm. The major advantage of this technique is the development of a solution algorithm, coupled with the multiple infinite trigonometric cum polynomial series solutions, leading to fast converging series solutions. A representative initial and boundary value problem governed by hyperbolic partial differential equations of variable coefficients is presented herein to demonstrate the efficiency and accuracy of the method.

INTRODUCTION

Exact solutions of boundary-value problems described by differential equations with variable coefficients can be obtained only for a limited class of problems. A similar situation exists if the boundary-value problem is formulated in an integral equation form. Indeed, even those problems that can be solved in exact closed form may preferably be treated through approximate or numerical methods, because evaluation of the exact solution may be much too complicated [1]. Among the most known classical numerical techniques are the variational methods as well as the method of weighted residuals (MWR) [2, 3]. Variational methods employ variational principles to obtain approximate solutions as continuous functions of a position in the media. However, use of a variational technique requires the existence of a functional which is not available for all classes of engineering mechanics problems [3, 4]. Whenever applicable variational techniques allow evaluation of an approximate solution through minimization of the appropriate functional.

Under the class of MWR, a wide variety of approximate techniques, such as the collocation, Galerkin's and Trefftz's methods are included. All MWR assume trial solutions usually in a polynomial form with undetermined coefficients. Each method uses a different approach to evaluate the coefficients through minimization of the error in a weighted form either within the domain or along the boundary of the system under consideration [5, 6]. Comparative studies on MWR as well as representative applications can be found in the text of Finlayson [7].

The classical approximate methods and the advent of powerful computers contributed to the emergence of finite-element, finite-difference and boundary-element methods [8–10]. Both finite-element and finite-difference methods determine unknown physical quantities by reducing the infinite degrees of freedom of a continuous system to a finite set amenable to computer-aided solutions. In particular, finite-difference techniques lead to a finite set of unknown quantities by defining a series of modes at which the discrete version of the differential equation is satisfied [8]. In finite-element methods, the differential equation is satisfied in an average sense over a region or an element [11]. The two techniques require discretization of the domain as well as the boundaries of the system under consideration [8, 9]. Boundary-element methods use interpolation functions that satisfy the governing equations in the domain but not the boundary conditions. Thus, they require discretization of the boundary only, and lead to significant reduction of computational efforts over domain techniques [10, 12]. Use of boundary-element methods requires the existence of appropriate fundamental solutions or Green's functions. Both boundary and domain methods lead to a large system of equations resulting from refined time and/or spatial discretizations.

It is well known that the classical analytical methods cannot be applied to solve boundary-value problems of irregular domains with general boundary conditions. However, the majority of the problems with arbitrary domains can be solved through numerical schemes as finite-element or finite-difference methods.

In a broader context of solving boundary-value problems of arbitrary shapes, discretization of a continuum with a minimum number of discrete elements is essential to minimize the number of unknowns and, if possible, satisfy compatibility along the boundaries of contiguous elements. Such an approach would not only reduce time and effort to obtain the solution, but improve the accuracy of final results, and would even minimize complexities while searching for eigenvalues of initial-value problems.

The solution scheme proposed herein is a product of undetermined coefficients and a sum of trigonometric and polynomial series. This scheme has been used to solve a wide range of initial as well as boundary-value problems. The assumed set of polynomials with undetermined constants is complete and a class of $(n-1)$, where n is the order of differential equation. This set is coupled with trigonometric series and forced to satisfy the boundary conditions to obtain the undetermined constants of the polynomial. Once the unknown variable coefficients of the given problem are expressed in trigonometric series, insertion of these series and the above-mentioned solution scheme into the governing partial differential equation, and application of orthogonality conditions, would lead to an infinite summation equation as a function of undetermined coefficients of the originally assumed solution scheme. This summation equation has been solved for these undetermined coefficients either in open form by considering the first few terms of the equation or in closed form through a special algorithm that has been derived herein, i.e. various steps of this derivations are shown in the Appendix. Examples are presented to formalize the above concepts of the proposed methodology.

Several advantages of the proposed methodology are:

(1) evaluation of response of physical systems with variable properties over their domains;
(2) minimization of number of finite elements in the domain;
(3) fast convergence of the summation equation that can satisfy arbitrary boundary conditions and be truncated at any desired level of accuracy;
(4) extrapolation of final results of higher accuracy through the established results for arbitrarily determined truncation levels;
(5) improved accuracy and computational efficiency through the proposed scheme without any loss of generalities of the physical system.

BASIC METHOD

For a well-posed initial boundary-value problem expressed in the form of a system of differential equations, the present approach assumes a trial solution comprised of Fourier series with undetermined coefficients and a complete set of polynomial series dependent on boundary and initial conditions. Evaluation of the undetermined coefficients is achieved by requiring the trial solution to satisfy the differential equation. Thus, an infinite set of simultaneous equations with the undetermined coefficients as unknowns is obtained. The proposed method is general, and can be employed to solve multi-dimensional initial boundary-value problems. However, the treatment of a one-dimensional engineering system under various dynamic loading conditions is presented herein to facilitate the understanding and attest to the accuracy and efficiency of the method.

Consider the following time dependent hyperbolic differential equation defined in a domain Ω

$$Aw = f \tag{1}$$

where A is a general linear differential operator involving spatial and time derivatives of an unknown function w, and f denotes a given function. On the boundary S of Ω, w will have to satisfy boundary and initial conditions.

Assume the following expression to be a trial solution of eqn (1)

$$w(x,t) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_k \sin\frac{k\pi x}{L}\sin\frac{l\pi t}{T} + \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij} x_j^{j-1} t_i^{i-1} \tag{2}$$

where the a_{kl} are undetermined coefficients; $k\pi x/L$, $l\pi t/T$ are known, linearly independent mode shapes or coordinate functions of x and t, respectively; b_{ij} are constants to be determined by satisfying the initial and boundary conditions; m and n are the number of initial and boundary conditions, respectively, associated with complete sets of polynomials; L is the length of a spatial domain and T is the period of the Fourier series expansion.

The coefficients b_{ij} can be expressed in terms of a_{kl} and the inhomogeneous boundary and initial quantities by satisfying eqn (2) with the initial and boundary conditions. Thus eqn (2) takes the form

$$w(x,t) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}\left[\sin\frac{k\pi x}{L} + \psi_k(x)\right] \times \left[\sin\frac{i\pi t}{T} + \phi_i(t)\right]. \tag{3}$$

The mode shapes within the brackets of eqn (3) satisfy the initial and boundary conditions, and are a set of complete and linearly independent functions over the domain Ω.

The method proceeds by expanding the variable coefficients of operator A in half-range Fourier series. Then substitution of the Fourier series expansion of variable coefficients and the assumed solution into eqn (1) as well as application of orthogonality conditions leads to a summation equation in terms of the undetermined coefficients, a_{kl}.

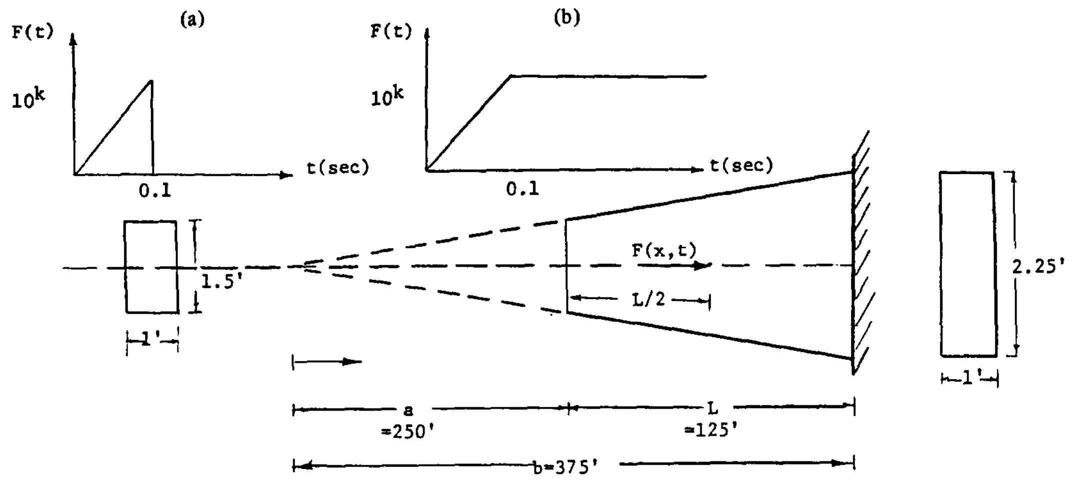

Fig. 1. Tapered beam of rectangular section with dynamic loads.

$$a_{kl}+\sum_{i=j}^{\infty}\sum_{j=1}^{\infty}B_{ijkl}a_{ij}=\phi_{kl} \tag{4}$$

where B_{ijkl} and ϕ_{kl} are known functions of the Fourier expansions of variable coefficients, boundary conditions, and the function, f, of eqn (1).

Accuracy of the method depends on the number of terms that are summed in the infinite series expansion (3). Thus, improvement of solution accuracy is achieved without redefining the mode shapes, incorporating additional shape function or employing a more refined discretization of the domain Ω, as is required by the finite-element method.

It should be mentioned that a_{kl} may be obtained by any weighted residual method that operates on the form given in eqn (3). Variational methods have also been used to find a_{kl} depending on the class of problems under investigation [6]. However, it is shown herein that through simple orthogonalization techniques, a set of simultaneous equations of fast convergence can be obtained while solving a differential equation as defined in eqn (1). A considerably simpler development of such a technique has been presented by the authors for steady-state problems [14].

EXAMPLES

To highlight the most important features of the method, the forced vibration phenomena of a rod with variable cross-sectional area subjected to transient loads (Fig. 1) are studied. Under the assumption of small deformation theory and linear elastic behavior, the axial response, $u(x,t)$, of a rod with linearly varying cross-sectional area (Fig. 1) is described by the hyperbolic differential equation with two independent variables x and t:

$$\frac{\partial^2 u}{\partial x^2}+\frac{1}{x}\frac{\partial u}{\partial x}-\frac{1}{C_0^2}\frac{\partial^2 u}{\partial t^2}=\frac{-F(x,t)}{EA_x} \tag{5}$$

where $F(x,t)$ is the forcing function, ρ the mass density, E the modulus of elasticity, A_0 the cross-sectional area at the tip, $A_x=A_0x/a$ and $C_0^2=E/\rho$. The boundary and initial conditions for the above problem are assumed respectively as

$$\left.\frac{\partial u}{\partial x}\right|_{x=a}=0 \text{ and } \left.u\right|_{x=b}=0 \tag{6a}$$

$$u(x,0)=\frac{\partial u(x,0)}{\partial t}=0. \tag{6b}$$

The Fourier cum polynomial series solution in the form of eqn (3) that satisfies eqns (6) is

$$u(x,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}u_{ij}\left(\sin\frac{i\pi t}{T}-\frac{i\pi t}{T}\right)\times\left[\sin\frac{j\pi x}{L}+\frac{j\pi}{L}(b-x)\cos\frac{j\pi a}{L}\right]. \tag{7}$$

Inserting eqn (7) into eqn (5), expanding $1/x$ in half range sine series and applying orthogonality conditions yields a summation equation similar to eqn (4) for general loading as

$$u_{ij}\left[\left(\frac{i\pi}{C_0T}\right)^2-\left(\frac{j\pi}{L}\right)^2\right]-\frac{2\pi\cos i\pi}{iL^2}\sum_{k=1}^{\infty}ku_{kj}\pi j^2-\sum_{l=1}^{\infty}u_{kl}S_{jl}+\sum_{l=1}^{\infty}u_{il}\frac{\pi}{L^2}S_{jl}\times\left\{2\left(\frac{i\pi}{C_0T}\right)^2 l\pi\cos\frac{l\pi a}{L}\times\left(\frac{1}{j\pi}\cos\frac{j\pi a}{L}+\frac{1-\cos j\pi}{j^2\pi^2}\cdot\sin\frac{j\pi a}{L}\right)-\sin\frac{l\pi b}{L}\cos\frac{j\pi a}{L}\left(\frac{1-\cos j\pi}{j\pi}\right)\right\}=F_{ij} \tag{8}$$

where S_{jl} and $F_{i,j}$ are given explicitly in the Appendix.

Table 1. Maximum amplitudes at tip and center

	Prop. theory† $\times 10^{-4}$ in.		NASTRAN $\times 10^{-4}$ in.	
T	Center	Tip	Center	Tip
Load of Fig. 1b				
0.1	6.52	5.64		
0.115	7.01	6.06		
0.2	5.63	4.84		
0.3	5.33	4.84	8.102	8.103
0.5	3.39	2.91		
Load of Fig. 1c				
0.1	6.52	5.64		
0.2	8.86	7.59	8.102	8.103
0.5	9.92	8.50		

† These values are obtained for two terms of the series in eqn (8).

For the rod shown in Fig. 1c, which is subjected to an axial concentrated load at the middle, the above summation eqn (8) is solved in open form. The properties of the structure are characterized by a Young's modulus $E = 30{,}000$ ksi, length $L = 1500$ in. and mass density $(\rho) = 75 \times 10^{-7}$ K sec^2 in^4.

Response amplitudes at the tip and the point of application of the load of a tapered cantilever rod under two types of dynamic loads are given in Table 1. In addition, Table 1 contains results obtained from a finite-element analysis using NASTRAN. In the finite-element analysis, the tapered rod was discretized into eight uniform axial elements.

The results of Table 1 clearly indicate that the accuracy of the axial response amplitude of a tapered rod greatly depends on the period T of the Fourier series expansion. As can be observed, the response amplitude for a load of limited duration (Fig. 1b) is found to be very close to the results from the finite-element analysis (NASTRAN) when the period T of the Fourier series expansion is in the vicinity of the duration of the load. This improvement in accuracy can be explained from the fact that the modified period is very close to the zero slope of the response function. For this loading, a Fourier expansion with larger T leads to lower magnitudes of response amplitude and these errors can be avoided by accounting for many of the higher-order terms of the series given in the summation solution. Therefore, it appears that the transient response of the system can be accurately predicted by appropriately choosing the period of the Fourier series whereas the steady-state response can be obtained either by considering many terms of the Fourier series expansion or reducing the problem from a hyperbolic to an elliptic differential equation [14].

As can be seen in Table 1, the response amplitudes of the rod with variable cross-sectional area under the loading shown in Fig. 1c do not depend on the choice of period T. Gibb's phenomenon has been noted in the numerical evaluations for response amplitudes. Hence, utilization of correction factors such as the ones suggested by Lanczos [15] could improve the accuracy of the final results. Additional improvements in the accuracy of response amplitudes can be achieved by shifting the response to perturbate around a constant value, i.e. add or subtract it from the STATIC response and then evaluate the response amplitudes.

From the extensive numerical investigations conducted by the authors, the most critical step to obtain accurate response amplitudes is the choice of period T. Additional knowledge on the selection of T has to be pursued on a systematic theoretical basis.

CONCLUSIONS

A general method to solve initial boundary-value problems with variable properties and arbitrary boundaries is developed. Solutions are given in closed form, and obtained with the aid of Fourier cum polynomial series expansions. In order to elucidate the method and attest to its accuracy, the response of a rod with variable thickness subjected to transient loads is determined. The numerical evaluations are carried out by using a highly efficient algorithm for summation equations that are developed as a part of this research.

The proposed method is highly sensitive to the choice of the period T for loadings of limited time duration. However, the accuracy of response amplitudes does not greatly depend on T for dynamic loads of infinite duration. Further investigations on the appropriate selection of period T have to be systematically carried out along the lines suggested in this text.

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APPENDIX

Closed form series solution of a first-order summation equation

Consider the following first-order summation equation that typically results in the solution scheme of a partial differential equation of two independent variables based on Fourier cum polynomial series.

$$h_i + \sum_{i_1=1}^{\infty} A_{i_1 i} h_{i_1} = \phi_i \quad \text{for} \quad i = 1, (1), \infty \tag{A1}$$

where A_{i_1} and ϕ_i are given converging functions and the unknown to be solved from eqn (A1) is h_i.

Equation (A1) can be written as by changing i to i_1 and i_1 to i_2:

$$h_{i_1} + \sum_{i_2}^{\infty} A_{i_2 i_1} h_{i_2} = \phi_{i_1}. \tag{A2}$$

Sum eqn (A2) with respect to the dummy index i_1 after multiplying it with A_{i_1}

$$\sum_{i_1=1}^{\infty} A_{i_1 i} h_{i_1} + \sum_{i_1=1}^{\infty} A_{i_1 i} \sum_{i_2=1}^{\infty} A_{i_2 i_1} h_{i_2} = \sum_{i_1=1}^{\infty} A_{i_1 i} \phi_{i_1}. \tag{A3}$$

Solving for

$$\sum_{i_1=1}^{\infty} A_{i_1 i} h_{i_1}$$

from eqn (A3) and substituting into (A1) yields

$$h_i - \sum_{i_1=1}^{\infty} A_{i_1 i} \sum_{i_2=1}^{\infty} A_{i_2 i_1} h_{i_2} = \phi_i - \sum_{i_1=1}^{\infty} A_{i_1 i} \phi_{i_1}. \tag{A4}$$

The second term of eqn (A4) is solved for in the manner identical to the second term of eqn (A1) which is the first term of eqn (A3), i.e. summing on dummy indices i_1 and i_2 of eqn (A4) gives:

$$\begin{aligned}\sum_{i_1=1}^{\infty} A_{i_1 i} \sum_{i_2=1}^{\infty} A_{i_2 i_1} h_{i_2} & - \sum_{i_1=1}^{\infty} A_{i_1 i} \sum_{i_2=1}^{\infty} A_{i_2 i_1} \sum_{i_3=1}^{\infty} A_{i_3 i_2} \sum_{i_4=1}^{\infty} A_{i_4 i_3} H_{i_4} \\ &= \sum_{i_1=1}^{\infty} A_{i_1 i} \sum_{i_2=1}^{\infty} A_{i_2 i_1} \phi_{i_2} - \sum_{i_1=1}^{\infty} A_{i_1 i} \sum_{i_2=1}^{\infty} A_{i_2 i_1} \sum_{i_3=1}^{\infty} A_{i_3 i_2} \phi_{i_3}.\end{aligned} \tag{A5}$$

Solving for the second term of eqn (A4) from eqn (A5) gives:

$$\begin{aligned}h_i - \sum_{i_1=1}^{\infty} A_{i_1 i} \sum_{i_2=1}^{\infty} A_{i_2 i_1} \sum_{i_3=1}^{\infty} A_{i_3 i_2} \sum_{i_4=1}^{\infty} A_{i_4 i_3} h_{i_4} &= \phi_i - \sum_{i_1=1}^{\infty} A_{i_1 i} \phi_{i_1} + \sum_{i_1=1}^{\infty} A_{i_1 i} \sum_{i_2=1}^{\infty} A_{i_2 i_1} \phi_{i_2} \\ &\quad - \sum_{i_1=1}^{\infty} A_{i_1 i} \sum_{i_2=1}^{\infty} A_{i_2 i_1} \sum_{i_3=1}^{\infty} A_{i_3 i_2} \phi_{i_3}.\end{aligned} \tag{A6}$$

A similar mathematical operation is repeated several times to derive the following solution form of the summation eqn (A1)

$$h_{i_0} - \left(\prod_{n=1}^{\infty} \sum_{i_n=1}^{\infty} A_{i_n i_{n-1}}\right) h_{i_n} = \phi_{i_0} + \sum_{N=1}^{\infty} (-1^N) \left(\prod_{n=1}^{\infty} \sum_{i_n=1}^{\infty} A_{i_n i_{n-1}}\right) \phi_{i_n} \tag{A7}$$

where i_0 is a dummy index varying from 1 to ∞.

Equation (A7) can be rewritten, for a reasonably large number of i_n values and for converging $A_{i_n i_{n-1}}$ and ϕ_{i_n}, as

$$h_{i_0} = \phi_{i_0} + \sum_{N=1}^{\bar N} (-1)^N \left(\prod_{n=1}^{N} \sum_{i_n=1}^{\infty} A_{i_n i_{n-1}}\right) \phi_{i_n}. \tag{A8}$$

Equation (A8) represents the final solution form of the first-order summation equation given in eqn (A1).

Depending upon the desired degree of accuracy, the series of eqn (A8) can be truncated by considering only the first few of its terms. Typically, $\bar N$ of 3 or 4 would be more than adequate to obtain results of high accuracy. If one needs the value of h_{50} for example, a (50 × 50) matrix has to be inverted through open form techniques. However, from eqn (A8), h_{50} can be routinely obtained by fixing a value of $\bar N$ as 3 or 4 and inserting i_0 as 50. Finally, the total number of multiplications in solving for h_{i_0} through eqn (A8) can be very small when compared to Gaussian elimination technique.

The S_{jl} and $F_{i,j}$ of eqn (A8) can be evaluated from:

$$S_{jl} = \left[\int_{(j+l)\pi a/L}^{(j+l)\pi b/L} \frac{\sin \bar x}{\bar x} + \int_{(j-l)\pi a/L}^{(j-l)\pi b/L} \frac{\sin \bar x}{\bar x} - 2\cos\frac{j\pi a}{L} \int_{l\pi a/L}^{l\pi b/L} \frac{\sin \bar x}{\bar x}\right] \mathrm{d}x$$

and

$$F_{i,j} = \frac{4F_0 \sin\frac{j\pi}{L}\left(\frac{L}{2}+a\right)}{TLt_0EA_0\left(1+\frac{L}{2a}\right)} \left(\frac{T}{i\pi}\right)^2 \sin\frac{i\pi t_0}{T} - \frac{t_0 T}{i\pi}\cos\frac{i\pi t_0}{T}$$

loading corresponding to Fig. 1b

$$= \frac{4F_0 \sin\frac{j\pi}{L}\left(\frac{L}{2}+a\right)}{TLt_0EA_0\left(1+\frac{L}{2a}\right)} \left(\frac{T}{i\pi}\right)^2 \sin\frac{i\pi t_0}{T} - \frac{t_0 T}{i\pi}\cos i\pi$$

loading corresponding to Fig. 1c.