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DYNAMIC RESPONSE OF FRAMEWORKS BY FAST FOURIER TRANSFORM

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Abstract—A general numerical method for determining the dynamic response of linear elastic plane frameworks to dynamic shocks, wind forces or earthquake excitations is presented. The method consists of formulating and solving the dynamic problem in the frequency domain by the finite element method and of obtaining the response by a numerical inversion of the transformed solution with the aid of the fast Fourier transform algorithm. The formulation is based on the exact solution of the transformed governing equation of motion of a beam element and it consequently leads to the exact solution of the problem. Flexural, and axial motion of the framework members are considered. The effects of damping (external viscous or internal viscoelastic), axial forces on bending, rotatory inertia and shear deformation on the dynamic response are also taken into account. Numerical examples to illustrate the method and demonstrate its advantages over other methods are presented.

1. INTRODUCTION

The conventional finite element method for determining the dynamic response of linear elastic frameworks is based on the lumped or consistent mass representation and on displacement functions which are solutions of the static governing equations of a beam element and it consequently leads to an approximate solution of the problem[1]. The dynamic stiffness approach assumes a continuous distribution of mass and utilizing a displacement function which is the exact solution of the governing equation of harmonic motion of a beam element provides the exact response of frameworks to harmonic excitations (e.g. [1, 2]). If the dynamic forces are of a general transient nature, the dynamic stiffness method in conjunction with modal analysis can be used to provide the dynamic response[3, 4]; however, this approach requires prior knowledge of natural frequencies and modal shapes. Application of Laplace transform with respect to time on the governing equation of motion of a beam element enables one to solve the general forced vibration problem by the Laplace transformed dynamic stiffness method and then numerically invert the solution to obtain the time-domain dynamic response. Thus, this approach retains all the advantages of the dynamic stiffness method (continuous distribution of mass, exact dynamic response) and does not require a knowledge of natural frequencies and modal shapes. Beskos and Boley[5], Manolis and Beskos[6, 7] and Beskos and Narayanan[8], successfully employed this method to determine the dynamic response of frameworks in conjunction with various methods of numerical Laplace transform inversion. However, as it was demonstrated in Narayanan and Beskos[9], among the existing numerical Laplace transform inversion algorithms those that are characterized by high accuracy are time consuming, while those that are computationally efficient exhibit an accuracy unacceptable in structural dynamics. In an effort to overcome this problem and develop a transformed dynamic stiffness method in conjunction with an accurate and efficient numerical inversion algorithm, Narayanan and Beskos[10] replaced Laplace transform by Fourier transform in the dynamic stiffness approach and thus they were able to utilize the accurate and efficient Fast Fourier Transform (FFT) algorithm of Cooley and Tukey[11] to perform the numerical inversion of the transformed solution. However, they faced serious difficulties in connection with the inversion problem due to the presence of sharp discontinuities with jumps extending from $-\infty$ to $+\infty$ in the transformed solution at those values of the frequency which coincided with the natural frequencies of the structure which was assumed to be undamped.

In this paper, consideration of damping (external viscous or internal viscoelastic) considerably ameliorates the situation. In addition, this paper considers not only flexural beam motion as in [10], but axial motion as well. Furthermore, the effects of axial forces on bending, rotatory inertia and shear deformation on the dynamic response are taken into account and the treatment of seismic forces is also described. Thus, the present work, can be considered as an improvement and generalization of the work of Narayanan and Beskos[10].

Use of the (FFT) algorithm has been witnessed in various areas of structural mechanics during the past decade. Its most extensive use has been in earthquake engineering, especially in connection with the problem of soil-structure interaction (e.g. [1, 12–16]). Other applications of the (FFT) algorithm include wave propagation in anisotropic plates[17, 18], and determination of the dynamic response of discrete structures by the frequency response method to random[19, 20] or deterministic disturbances[21, 22]. With the exception of Ref. [22], which presents just an example of a simply supported beam with a continuous distribution of mass without any theory permitting any generalizations, frameworks are treated as discrete systems in the above references. Narayanan and Beskos[10] were the first to employ (FFT) for the dynamic analysis of plane frameworks considered as structures composed of beams with a continuous mass distribution and thus obtain the exact solution of the problem. The present paper improves and generalizes their methodology.

2. TRANSFORMED DYNAMIC STIFFNESS MATRICES

Consider a uniform linear elastic beam element 1–2 of length L (Fig. 1) undergoing axial and flexural motions

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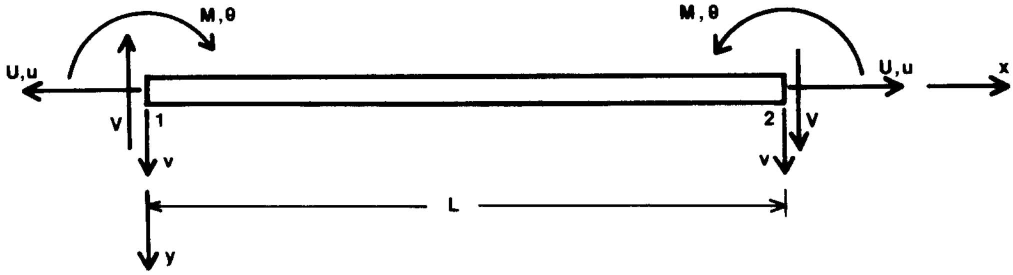

Fig. 1. Positive beam displacements and forces in mechanics convention.

which, on the basis of the Bernoulli–Euler beam theory, are governed, respectively, by the following uncoupled equations:

$$EAu'' - m\ddot{u} = 0,$$
$$EIv'''' + m\ddot{v} = 0, \quad (1)$$

where $u = u(x, t)$ and $v = v(x, t)$ are the axial and lateral displacements of the beam, respectively, E is the modulus of elasticity, m is the mass per unit length of the beam, A and I are the area and moment of inertia of the cross section of the beam, respectively, primes indicate differentiation with respect to the distance x along the length of the beam and dots indicate differentiation with respect to the time t.

The Fourier transform $\bar{y}(\omega)$ of a function $y(t)$ is defined by

$$\bar{y}(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-i\omega t}\, dt, \quad (2)$$

where ω is the circular frequency and $i = \sqrt{-1}$. Application of Fourier transform with respect to time on eqns (1) yields

$$\bar{u}'' + K_1{}^2\bar{u} = 0,$$
$$\bar{v}'''' - K^4\bar{v} = 0, \quad (3)$$

where

$$K_1{}^2 = m\omega^2/EA,$$
$$K^4 = m\omega^2/EI. \quad (4)$$

The solution of eqns (3) takes the form

$$\bar{u}(x, \omega) = D_1 \sin K_1x + F_1 \cos K_1x,$$
$$\bar{v}(x, \omega) = D_2 \sin Kx + F_2 \cos Kx + D_3 \sinh Kx + F_3 \cosh Kx, \quad (5)$$

where D_1, F_1, D_2, F_2, D_3 and F_3 are constants. Consider the beam element 1-2 of Fig. 2 with nodes 1 and 2, which has one and two degrees of freedom per node for axial and flexural transformed motions, respectively. Figure 2 shows the positive directions of the nodal displacements and the corresponding nodal forces for the two kinds of motion in the frequency domain. Evaluation of $\bar{u}$, $\bar{v}$ and $d\bar{v}/dx$ at the nodes 1 and 2 permits one to express the displacement functions (5) in terms of nodal displacements as

$$\bar{u}(x, \omega) = \{N_1(x, \omega)\}^T\{\bar{u}_1\bar{u}_2\},$$
$$\bar{v}(x, \omega) = \{N(x, \omega)\}^T\{\bar{v}_1\bar{\theta}_1\bar{v}_2\bar{\theta}_2\}, \quad (6)$$

where $\{N_1\}$ and $\{N\}$ are the vectors of the shape functions and T denotes transposition. Following standard finite element procedures (e.g. [23]) one can determine the element dynamic stiffness matrices in the frequency domain from

$$[D'] = \int_0^L \{N_1'\}\{N_1'\}^T EA\, dx, \quad (7)$$

$$[\bar{D}] = \int_0^L \{N''\}\{N''\}^T EI\, dx, \quad (8)$$

for axial and flexural motion, respectively. Thus, with the sign convention of Fig. 2, the following element transformed nodal force-displacement relations in terms of the $\bar{D}ij$ coefficients result:

$$\begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \end{bmatrix} = \begin{bmatrix} \bar{D}'_{11} & \bar{D}'_{12} \\ \bar{D}'_{21} & \bar{D}'_{22} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_s \end{bmatrix} \quad (9)$$

for the axial motion, and

$$\begin{bmatrix} \bar{V}_1 \\ \bar{M}_1 \\ \bar{V}_2 \\ \bar{M}_2 \end{bmatrix} = \begin{bmatrix} \bar{D}_{11} & \bar{D}_{12} & \bar{D}_{13} & \bar{D}_{14} \\ \bar{D}_{21} & \bar{D}_{22} & \bar{D}_{23} & \bar{D}_{24} \\ \bar{D}_{31} & \bar{D}_{32} & \bar{D}_{33} & \bar{D}_{23} \\ \bar{D}_{41} & \bar{D}_{42} & \bar{D}_{43} & \bar{D}_{44} \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{bmatrix} \quad (10)$$

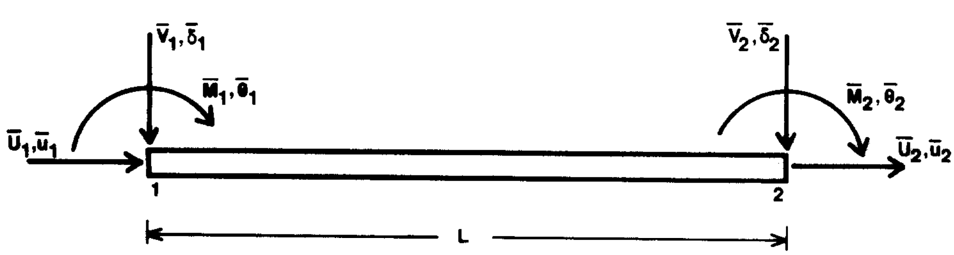

Fig. 2. Positive beam nodal transformed displacements and forces.

for the flexural motion, where

$$\bar{D}'_{11} = \bar{D}'_{22} = AEK_1 \cot(K_1L), \quad (11)$$
$$\bar{D}'_{12} = \bar{D}'_{21} = -AEK_1 \operatorname{cosec}(K_1L)$$

$$\bar{D}_{11} = \bar{D}_{33} = QK^2(sch + csh),$$
$$\bar{D}_{12} = \bar{D}_{21} = -\bar{D}_{34} = -\bar{D}_{43} = QKssh,$$
$$\bar{D}_{13} = \bar{D}_{31} = -QK^2(s + sh),$$
$$\bar{D}_{14} = \bar{D}_{41} = -\bar{D}_{23} = -\bar{D}_{32} = QK(ch - c), \quad (12)$$
$$\bar{D}_{22} = \bar{D}_{44} = Q(sch - csh),$$
$$\bar{D}_{24} = \bar{D}_{42} = Q(sh - s),$$
$$Q = EIK/(1 - cch)$$
$$s = \sin KL,\ c = \cos KL,\ sh = \sinh KL,\ ch = \cosh KL,$$

and where K_1, K_2 and K are given by (4).

3. EFFECT OF DAMPING

In this paper both internal viscoelastic damping and external viscous damping are considered. Internal viscoelastic damping is taken into account by assuming, for reasons of simplicity, that the beam material is a Kelvin solid obeying the constitutive law

$$\sigma = E(1 + f d/dt)\epsilon, \quad (13)$$

where σ is the stress, ϵ is the strain and f is the damping coefficient. More general viscoelastic models as described, for example, in Boley and Weiner[24] could also have been used without any particular difficulty. Equation (13) in the frequency domain becomes

$$\bar{\sigma} = E(1 + i\omega f)\bar{\epsilon}, \quad (14)$$

indicating that the formulation of the internally damped beam problem can be obtained from the undamped one be simply replacing E by $E(1 + i\omega f)$ in the frequency domain. For a large class of materials the relative loss of energy (damping) is independent of frequency for a wide range of frequencies implying that the term $i\omega f$ in (14) has to be replaced by if for these materials[25].

When external viscous damping is present, it is accounted for in the displacement equation of the beam motion by a damping force per unit length R proportional to the velocity and opposing the motion, i.e.

$$R = -c(dz/dt), \quad (15)$$

where c is the coefficient of damping and z may be either the u or the v displacement. Application of Fourier transform on (15) yields

$$\bar{R} = -i\omega c\bar{z}, \quad (16)$$

indicating that when external damping is present, eqns (3) remain the same while eqns (4) are replaced by

$$K_1^2 = (m\omega^2 - i\omega c)/AE,$$
$$K^4 = (m\omega^2 - i\omega c)/EI. \quad (17)$$

It should be mentioned that the above formulation permits damping (internal or external) to vary from member to member of the framework and is clearly a more rational way to represent damping properties than the conventional way based on modal critical damping values. Besides this representation permits more effective response control by taking advantage of the freedom of varying the damping of a large number of elements.

4. EFFECT OF AXIAL FORCE ON BENDING

In the presence of a constant compressive force P, eqn $(1)_2$ is replaced by

$$EIv'''' + Pv'' + m\ddot{v} = 0, \quad (18)$$

which becomes

$$\bar{v}'''' + 2\mu\bar{v}'' - \nu\bar{v} = 0 \quad (19)$$

in the frequency domain where

$$2\mu = P/EI,\ \nu = m\omega^2/EI. \quad (20)$$

The solution of (19) is

$$\bar{v}(x, \omega) = A \sinh \lambda_1 x + B \cosh \lambda_1 x + C \sin \lambda_2 x + D \cos \lambda_2 x, \quad (21)$$

where A, B, C and D are constants of integration and

$$\lambda_1 = (-\mu + \sqrt{(\mu^2 + \nu)})^{1/2},\ \lambda_2 = (\mu + \sqrt{(\mu^2 + \nu)})^{1/2}. \quad (22)$$

On the basis of the displacement function (21) and following the procedure described in Section 2, one can write the nodal force-displacement relation for a beam-column in the frequency domain exactly as in (10) with

$$\bar{D}_{11} = \bar{D}_{33} = -Q\sqrt{\nu}r(\lambda_2 sch + \lambda_1 csh),$$
$$\bar{D}_{12} = \bar{D}_{21} = -\bar{D}_{34} = -\bar{D}_{43} = 2Q\sqrt{\nu}[\mu(cch - 1) - \sqrt{\nu}ssh],$$
$$\bar{D}_{13} = \bar{D}_{31} = Q\sqrt{\nu}r(\lambda_1 sh + \lambda_2 s),$$
$$\bar{D}_{14} = \bar{D}_{41} = -\bar{D}_{23} = \bar{D}_{32} = Q\sqrt{\nu}r(c - ch), \quad (23)$$
$$\bar{D}_{22} = \bar{D}_{44} = Qr(\lambda_2 csh - \lambda_1 sch),$$
$$\bar{D}_{24} = \bar{D}_{42} = Qr(\lambda_1 s - \lambda_2 sh),$$
$$Q = EI/2\sqrt{\nu}(cch - 1) + 2\mu ssh,\ r = 2\sqrt{(\mu^2 + \nu)},$$
$$s = \sin \lambda_2 L,\ c = \cos \lambda_2 L,\ sh = \sinh \lambda_1 L, ch = \cosh \lambda_2 L.$$

The case of a tensile axial force P can be very easily taken into account by simply replacing P by $-P$ in all the previous expressions.

The combined damping and beam-column effects can also be treated on the basis of eqns (10) and (23) either by replacing E by $E(1 + i\omega f)$ or $E(1 + if)$ for the case of internal viscoelastic damping or by defining ν in (20) as $\nu = (m\omega^2 - i\omega c)/EI$ for the case of external viscous damping.

5. EFFECT OF ROTATORY INERTIA AND SHEAR DEFORMATION

The governing equations for lateral vibrations of Timoshenko beams, i.e. beams for which the effects of rotatory inertia and shear deformation are taken into account, are[26].

$$EI\psi'''' + m\ddot{\psi} - \rho I[1 + (E/sG)]\ddot{\psi}'' + (\rho^2 I/sG)\ddddot{\psi} = 0,$$
$$EIy'''' + m\ddot{y} - \rho I[1 + (E/sG)]\ddot{y}'' + (\rho^2 I/sG)\ddddot{y} = 0, \quad (24)$$

where s is the shear coefficient, G is the shear modulus, ρ is the mass density, $y = y(x, t)$ is the total lateral deflection and $\psi = \psi(x, t)$ is the bending slope of the beam element of length L under end forces only. Application of Fourier transform with respect to time on eqns (24) yields

$$\bar{\psi}'''' + 2\mu_0\bar{\psi}'' + \bar{\nu}_0\bar{\psi} = 0,$$
$$\bar{y}'''' + 2\mu_0\bar{y}'' + \nu_0\bar{y} = 0, \quad (25)$$

where

$$2\mu_0 = \rho[1 + (E/sG)]\omega^2/E,$$
$$\nu_0 = [(\rho^2 I/sG)\omega^4 - m\omega^2]/EI. \quad (26)$$

The general solutions of (25) can be expressed in terms of hyperbolic and trigonometric functions for the two cases of $\nu_0 > 0$ and $\nu_0 < 0$ and lead through standard procedures to the construction of $\bar{D}ij$ coefficients for a Timoshenko beam. This can be done, for example, by following Ref. [2, 27]. However, this approach lacks generality and leads to very complicated expressions, especially when other effects, such as damping, are also taken into account. A general and compact way of treating (25) is to write their solutions in the form

$$\bar{y}(x, \omega) = c_1 e^{\xi_1 x} + c_2 e^{\xi_2 x} + c_3 e^{\xi_3 x} + c_4 e^{\xi_4 x},$$
$$\bar{\psi}(x, \omega) = c_1' e^{\xi_1 x} + c_2' e^{\xi_2 x} + c_3' e^{\xi_3 x} + c_4' e^{\xi_4 x}, \quad (27)$$

where c_1, c_2, c_3, c_4, c_1', c_2', c_3', and c_4' are constants and ξ_1, ξ_2, ξ_3 and ξ_4 are in general complex numbers given as

$$\xi_{1,2} = \pm(-\mu_0 + \surd(\mu_0{}^2 - \nu_0))^{1/2},$$
$$\xi_{3,4} = \pm(-\mu_0 - \surd(\mu_0{}^2 - \nu_0))^{1/2}. \quad (28)$$

On the basis of the displacement functions (27) and following the procedure of Refs. [7, 8] semi-explicit expressions for the $\bar{D}ij$ coefficients can be obtained which can be found in[7]. The complete computation of these complicated coefficients is done numerically on the computer.

The effect of a constant axial force on bending can be very easily included in this formulation by adding the term P/EI to the r.h.s. of (26), as an inspection of (19), (20) and (25), (26) can reveal.

Internal viscoelastic damping of the Kelvin solid type can be very easily taken into account by simply replacing E and G by $E(1 + i\omega f)$ or $E(1 + if)$ and $G(1 + i\omega f)$ or $G(1 + if)$, respectively, in this formulation in view of (14) and on the assumption that there is no coupling between bending and shear deformation with respect to internal damping. A more rigorous way of taking into account internal viscoelastic damping is by employing the correspondence principle[24] and observing that one can go from the Laplace to the Fourier domain by replacing the Laplace transform parameter s by $i\omega$. Thus, for a Kelvin solid described by $s_{ij} = G(e_{ij} + f\dot{e}_{ij})$, where s_{ij} and e_{ij} are the diviatoric components of stress and strain tensors, respectively, this implies replacing E and G in the frequency domain by

$$E' = E(1 + i\omega f)/[1 + (1 - (E/3G))i\omega f],$$
$$G' = G(1 + i\omega f), \quad (29)$$

respectively. External viscous damping can be taken into account in the present formulation by including in (24) the effect of a distributed load $R = -c\dot{y}$ with the aid of eqn (27) of Ref. [27]. Equations (25) then take the form

$$\bar{\psi}'''' + 2\mu_0\bar{\psi}'' + \nu_0\bar{\psi} = -ci\omega\bar{y}',$$
$$\bar{y}'''' + 2\mu_0^*\bar{y}'' + \nu_0^*\bar{y} = 0, \quad (30)$$

where

$$2\mu_0^* = 2\mu_0 - (ic\omega/sGA),$$
$$\nu_0^* = \nu_0 - i(\rho c/sGAE)\omega^3 + i(c/EI)\omega. \quad (31)$$

Equation $(30)_2$ can yield a solution for $\bar{y}$ which is of the same form as $(27)_1$ and from which $\bar{y}'$ needed in $(30)_1$ can be evaluated. This permits to obtain $\bar{\psi}$ from $(30)_1$ as the sum of complementary and a particular solution both of the form of (27). Use of the displacement functions $\bar{y}$ and $\bar{\psi}$ permits one to construct the appropriate $\bar{D}ij$ coefficients with the aid of Ref. [7] numerically on the computer. Thus, it has been demonstrated in this section that it is possible to construct $\bar{D}ij$ coefficients for a beam element for the most general case which includes bending, effect of axial force, shear deformation and rotatory inertia, as well as damping (internal viscoelastic or external viscous).

6. FORMULATION AND SOLUTION OF THE PROBLEM

With the aid of the various $\bar{D}ij$ coefficients previously defined, the dynamic problem of a general plane framework can be expressed in the frequency domain in the static-like form

$$\{\bar{F}(\omega)\} = [\bar{D}(\omega)]\{\bar{v}(\omega)\}, \quad (32)$$

where $\{\bar{F}(\omega)\}$ and $\{\bar{v}(\omega)\}$ are the Fourier transformed dynamic load and displacement vectors, respectively, while $[\bar{D}(\omega)]$ is the transformed dynamic stiffness matrix of the structure obtained by an appropriate superposition of element stiffness matrices. After application of the transformed boundary conditions, the transformed solution $\{\bar{v}(\omega)\}$ of (32) is obtained, for a sequency of values of ω, by a standard matrix inversion in numerical form. From this solution the response $\{v(t)\}$ is obtained, for a sequence of values of t, by a numerical inversion of the Fourier transformed displacement vector. The solution thus obtained is the exact solution of the dynamic problem because the transformed dynamic element stiffnesses have been constructed on the basis of displacement functions which are the exact solutions of the transformed equations of motion. In most of the practical cases the components of the force vector are complicated functions of time and their direct Fourier transform has to be computed numerically. Both the direct and the inverse numerical Fourier transforms are discussed in detail in the next section.

The above formulation of the problem is based on the assumption of zero initial conditions. If there are nonzero initial conditions then the response of the structure can be obtained as the superposition of responses corresponding to a free vibration problem under non-zero initial conditions and to a forced vibration problem under zero initial conditions. This is clearly a disadvantage of the method which cannot handle initial conditions in one step as, for example, the similar method based on the Laplace transform[7].

When a framework is subjected to seismic forces, inertia forces are developed on its various members which have to be taken into account in the formulation of the problem. This will be illustrated for the case of the flexural motion. Consider a vertical beam element in flexural motion due to seismic horizontal forces characterized by the earthquake acceleration $\ddot{v}_s(t)$ as shown in Fig. 3(a). In this case eqn $(1)_2$ takes the form

$$EIv_r'''' + m\ddot{v}_r = -m\ddot{v}_s, \tag{33}$$

where the relative displacement $v_r = v_r(x, t)$, the absolute displacement $v = v(x, t)$ and the ground displacement $v_s = v_s(t)$ are related by

$$v_r = v - v_s. \tag{34}$$

Application of the Fourier transform with respect to time on (33) yields

$$\bar{v}_r'''' - 4K^4\bar{v}_r = \bar{p}(\omega)$$
$$\bar{p}(\omega) = -(m/EI)F[\ddot{v}_s],\ K^4 = m\omega^2/4EI, \tag{35}$$

where $F[\ddot{v}_s]$ is the Fourier transform of the earthquake acceleration. Equation (35) indicates that inertia effects are represented for a member i in the frequency domain by a uniformly distributed load $\bar{p}(\omega)$ which can be converted to equivalent inertia nodal forces and moments $\bar{F}_j^i$, $\bar{M}_j^i$, $j = 1, 2$, shown in Fig. 3(a) and given by

$$\bar{F}_1{}^i = \bar{F}_2{}^i = \bar{p}(\omega)L/2,$$
$$\bar{M}_1{}^i = \bar{M}_2{}^i = \bar{p}(\omega)L^2/12. \tag{36}$$

The inertia forces acting at the two nodes of a horizontal beam element are simply given by

$$\bar{T}_1{}^i = \bar{T}_2{}^i = (mL/2)(-\omega^2\bar{v}_r + F[\ddot{v}_s]) \tag{37}$$

and shown in Fig. 3(b). Thus, when a framework is subjected to seismic forces the problem can be described by (32) with the vectors $\{\bar{F}(\omega)\}$ and $\{\bar{v}(\omega)\}$ representing Fourier transformed nodal inertia forces and relative displacements, respectively.

7. THE FAST FOURIER TRANSFORM

The two most important computational problems of the proposed method for determining the dynamic response of frameworks with a continuous distribution of mass are the numerical evaluation of direct and inverse Fourier transforms. The Fourier transform pair of two

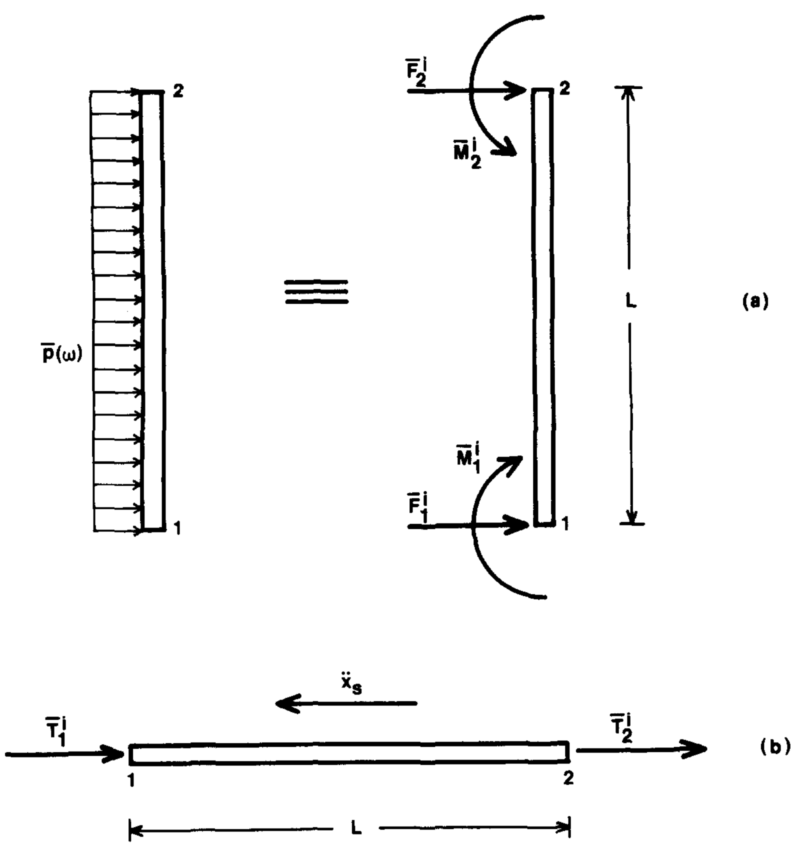

Fig. 3. Equivalent nodal inertia forces and moments for a horizontal and a vertical beam element.

functions $y(t)$ and $\bar{y}(\omega)$ is defined by the reciprocal relations

$$\bar{y}(\omega) = \int_{-\infty}^{\infty} y(t)\,e^{-i\omega t}\,dt,$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{y}(\omega)\,e^{i\omega t}\,d\omega, \quad (38)$$

where the relation $\omega = 2\pi\phi$ connects the circular frequency ω with the frequency ϕ. The function $\bar{y}(\omega)$ in $(38)_1$ is called the direct Fourier transform of $y(t)$, while the function $y(t)$ in $(38)_2$ is called the inverse Fourier transform.

In many engineering problems which involve complicated functions or sampled data, use of numerical techniques is imperative and this requires the functions to be given in discrete form. The Fourier transform pair of (38) for functions $y(t)$ which are zero for $t < 0$ can be written in discrete form as [28, 29]

$$\bar{y}\left(\frac{n}{NT}\right) = T\sum_{\lambda=0}^{N-1} y(\lambda T)\,e^{-i2\pi n\lambda/N}$$

$$y(\lambda T) = \frac{F}{2\pi}\sum_{n=0}^{N-1} \bar{y}\left(\frac{n}{NT}\right)e^{i2\pi n\lambda/N}, \quad (39)$$

where $\lambda, n = 0, 1, 2, \ldots, N-1$, T is the sampling interval, i.e. the distance between two successive discrete points in the time domain and $NT = F$ is the frequency interval. The above discrete Fourier transform pair relates N equally spaced samples $y(\lambda T)$ in the time domain to N equally spaced samples $\bar{y}(n/NT)$ in the frequency domain. The relationship between the discrete and continuous Fourier transforms is governed by the sampling theory (e.g. [29]).

Equation $(39)_2$ indicates that it is possible to obtain N samples of a time function from N samples of its Fourier transform. What is needed is to choose a number of samples N required for the time range T_0, which determines the frequency range $F_0 = N \cdot F = N/T_0$, and then to sample the Fourier transform at N equally spaced points between zero and F_0. The details of this process, the difficulties involved and the ways to overcome them for the particular subject of this paper are described in the next section.

A direct calculation of $(39)_1$ or $(39)_2$ as an accumulated sum of products for each n or λ, respectively, would require N^2 operations. The Fast Fourier Transform (FFT) is an algorithm for computing $(39)_1$ or $(39)_2$ in $N \log N$ operations, where "operation" means a complex multiplication and addition. The FFT algorithm is described in great detail in Ref. [29]. It has been programmed in Fortran language and there are subroutines available which perform this algorithm for real or complex data (e.g. [30]). In this paper use is made of the subroutine FOUR 1 of Ref. [30], which performs one-dimensional transforms on complex arrays whose lengths are powers of two.

8. STRUCTURAL EXAMPLES

The following numerical examples serve to illustrate the method and demonstrate its merits. All the numerical computations were performed on a CDC Cyber 74 computer.

Example 1

Consider the simply supported beam 1–3 subjected to a suddenly applied load $P(t)$ at its midspan as shown in Fig. 4. It is requested to determine the midspan deflection $\delta(t)$ of this beam for various amounts of internal viscoelastic damping of the Kelvin type.

Because of the small size of the structure, the formulation and solution of the problem in the frequency domain is done analytically. Because of the symmetry, only odd modes contribute to $\delta_2(t)$ and thus the rotation at the midspan is taken to be zero. Hence, eqn (32) applied to the beam element 1–2, after application of the boundary conditions takes the form

$$\bar{P}(\omega) = 2[\bar{D}_{33} - (\bar{D}_{23}^2/\bar{D}_{22})]\bar{\delta}(\omega), \quad (40)$$

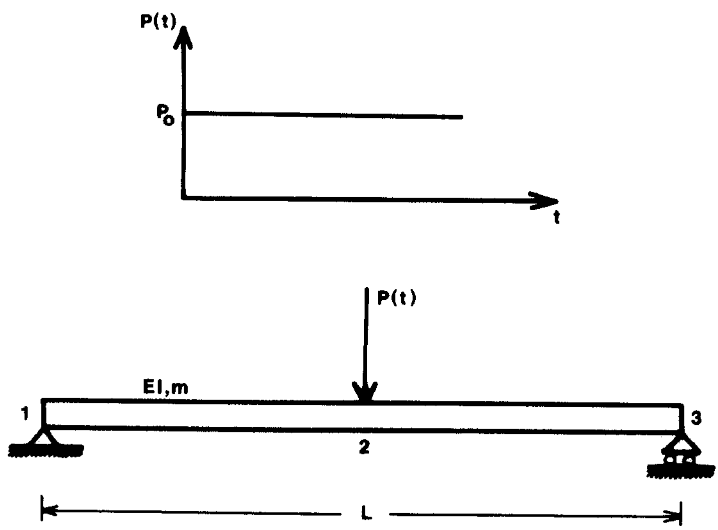

Fig. 4. The simply supported beam of Example 1.

where the $\bar{D}_{ij}$ coefficients are given by (12) and (4) and $\bar{P}(\omega) = P_0/i\omega$. Using (12) and solving (40) for $\bar{\delta}(\omega)$ yields

$$\bar{\delta}(\omega) = \frac{P_0}{8EI\mu^3}\bar{f}(\omega), \quad (41)$$

$$\bar{f}(\omega) = \frac{-i}{\omega^2\sqrt{(2\omega)}}\left(\tan\frac{KL}{2} - \tanh\frac{KL}{2}\right) \quad (42)$$

$$\mu = (m/4EI)^{1/4}, \; KL = \mu\sqrt{(2\omega)}L. \quad (43)$$

The function $\bar{\delta}(\omega)$ was written in the above form for reasons of comparison of the present results with those of Refs. 5 and 10 on the basis of the function $\bar{f}(\omega)$. For zero damping, $\bar{f}(\omega)$ is an imaginary and odd function of ω which exhibits discontinuities with jumps from $-\infty$ to $+\infty$ at those values of ω which are the natural frequencies of the beam corresponding to the odd modes which contribute to the midspan deflection. For nonzero internal damping E is replaced by $E(1 + i\omega f)$ in (43) and $\bar{f}(\omega)$ becomes a complex function of ω with real and imaginary parts plotted in Fig. 5 for the special case of the W 10×21 beam of Refs. 5 and 10 with $L = 12$ ft, $I = 106.3$ in^4, $w = 21$ lb/ft, $E = 30\times10^6$ psi and $f = 0.0001$. It is now apparent that $\bar{f}(\omega)$ does not have any discontinuities with infinite jumps and this permits one to accomplish the discrete representation of its real and imaginary parts more accurately and conveniently than in the case with zero damping. In fact one can approximately treat the zero damping case as a damped one with very small values of f. The error of this approximation can be negligible for small enough values of f as subsequent results indicate.

The process of numerical inversion of the function $\bar{f}(\omega)$ consists of the following three steps:

(i) Choosing the number of samples N and the frequency $\Omega_0 = 2\pi F_0$.

(ii) Sampling the function $\bar{f}(\omega)$ in the interval $(-\Omega_0, \Omega_0)$ using a sampling interval $\Omega = \Omega_0/N$.

(iii) Performing numerical inversion of this sampled function using the FFT algorithm to obtain the values of $f(t)$ at intervals $T = \pi/\Omega_0$ in the time domain.

The sampling process usually leads to errors due to aliasing and truncation[29]. The function $\bar{f}(\omega)$ can be truncated by taking the frequency range Ω_0 to be equal to some finite value of ω beyond which it is assumed that $\bar{f}(\omega)$ is zero, in order to conform with the sampling theorem[29]. It is well known that in framework dynamics only the first few modes significantly contribute to the response and this reflects to the fact that $\bar{f}(\omega)$, apart from discontinuities, decreases as ω increases. Thus the truncation error decreases for increasing Ω_0. The aliasing error can be reduced to an acceptable level by decreasing the sample interval Ω in the frequency domain. However, the interval Ω for which aliasing is small is also related and depends on the choice of Ω_0. Also, the number of samples N is limited to be a power of 2 for the program FOUR 1 of Ref. [30] which

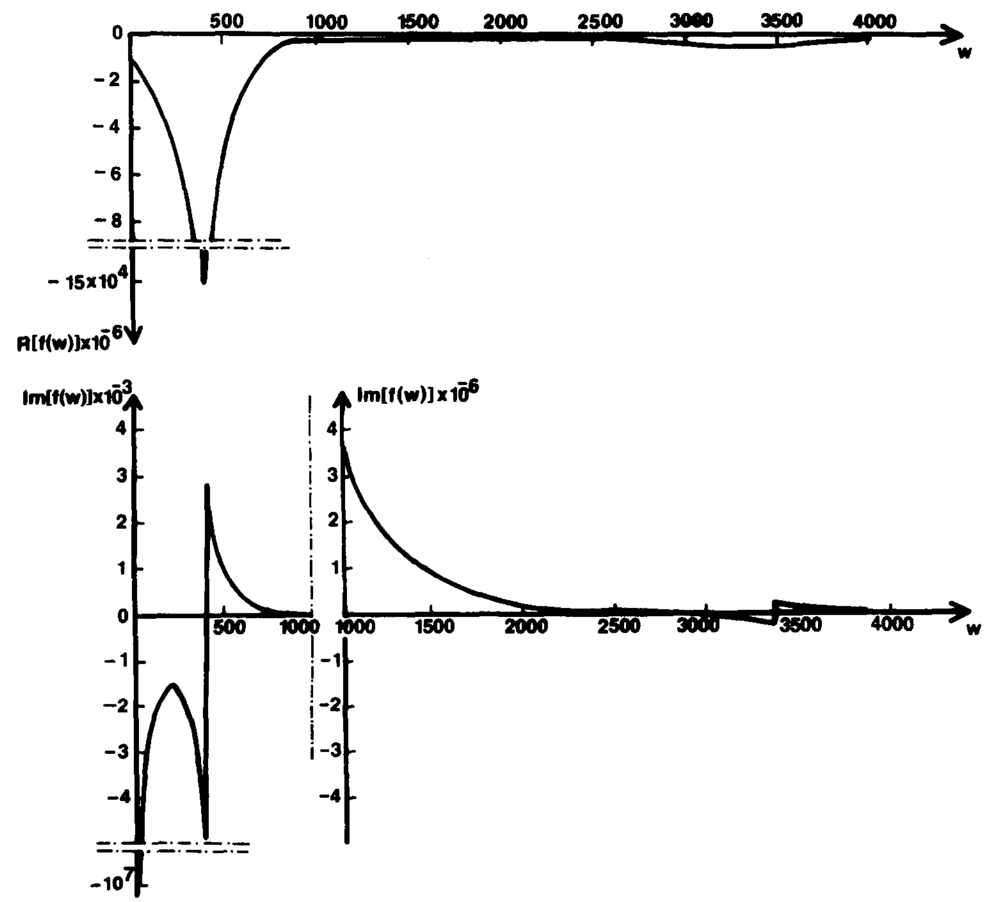

Fig. 5. Plot of the real and imaginary parts of $\bar{f}(\omega)$ vs ω.

was used in this work. The reasons for adopting FOUR 1 here are its simplicity and great speed. It should be noted that the definition of the Fourier pair in FOUR 1 is

$$\bar{y}(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(t)\,e^{-i\omega t}\,dt,$$

$$y(t) = \int_{-\infty}^{\infty} \bar{y}(w)\,e^{i\omega t}\,dt.$$

In this example by taking $N = 2^{10} = 1024$ and $\Omega_0 = 2500$ which give $\Omega = 2.441406$, the numerical inversion produces a function $f(t)$ given in discrete form, at intervals of length $T = \pi/\Omega_0 = 0.001256$ and for various values of the damping coefficient f, by Fig. 6. Taking into account that the real part of $\bar{f}(\omega)$ is even and the imaginary odd, the sampling was not done in the interval $(-\Omega_0, \Omega_0)$ but in the interval $(0, 2\Omega_0)$. The CP computer time spent on the inversion was 2.450 sec. For the value of $f = 10^{-8} \approx 0.0$ the results of the undamped case as reported in [5, 10] are recovered within plotting accuracy. Figure 6 clearly demonstrates the decreasing effect of damping on the dynamic response.

If initial conditions $v(L/2, 0) = v_0$ and $\dot{v}(L/2, 0) = \dot{v}_0$ were given then the total response would be the sum of the forced vibration response computed by Fourier transform and the

$$v(L/2, t) = \sum_{n=1,3,5}^{\infty}\left[\left(\dot{v}_0 \sum_{n=1,3,5}^{\infty} \omega_n \sin\frac{n\pi}{2}\right)\sin\omega_n t + \left(v_0 \sum_{n=1,3,5}^{\infty} \sin\frac{n\pi}{2}\right)\cos\omega_n t\right]\sin\frac{n\pi}{2}$$

obtained under free vibration conditions where $\omega_n = (n^2\pi^2/L^2)\sqrt{(EI/m)}$.

Example 2

Consider a steel portal frame subjected to lateral wind forces uniformly distributed and time dependent as shown in Fig. 7(a). The horizontal deflection of this frame at the level of the girder under the assumptions of negligible axial deformation and zero damping was determined in [6] by numerical Laplace transform. The response of the same structure is also computed here by numerical Fourier transform assuming nonzero damping and taking into account axial deformation.

On the assumption that the axial deformation is negligible the frame has only three free nodal displacements $(\bar{\delta}_2 = \bar{\delta}_3)$ and eqn (32) after application of the boundary conditions takes the form

$$\begin{Bmatrix} \bar{P}_2(\omega) \\ \bar{M}_2(\omega) \\ \bar{M}_3(\omega) \end{Bmatrix} = \begin{bmatrix} 2D_{33} - mL\omega^2 & -\bar{D}_{43} & -\bar{D}_{43} \\ -D\bar{D}_{43} & 2\bar{D}_{44} & \bar{D}_{42} \\ -\bar{D}_{43} & \bar{D}_{42} & 2\bar{D}_{44} \end{bmatrix} \begin{Bmatrix} \bar{\delta}_2(\omega) \\ \bar{\theta}_2(\omega) \\ \bar{\theta}_3(\omega) \end{Bmatrix} \tag{44}$$

where

$$\begin{aligned} \bar{P}_2(\omega) &= (q_0L/2)[(-i/\omega) + (2/\omega^2)], \\ \bar{M}_2(\omega) &= -(q_0L^2/12)[(-i/\omega) + (2/\omega^2)], \\ \bar{M}_3(\omega) &= 0. \end{aligned} \tag{45}$$

The solution of (44) for the transformed displacement vector was done numerically for a sequence of values of ω and the response vector was finally obtained in the time domain by a numerical inversion of the transformed solution utilizing the FFT algorithm with $N = 512$ and $\Omega_0 = 25.31$. The numerical data pertaining to this problem consists of $L = 12$ ft., $q_0 = 20$ psi, $E = 29 \times 10^6$ psi, $I = 395$ in.^{4}, $w = 50$ lb/ft and $f = 10^{-6} \approx 0$, 10^{-4} and 10^{-3}. The

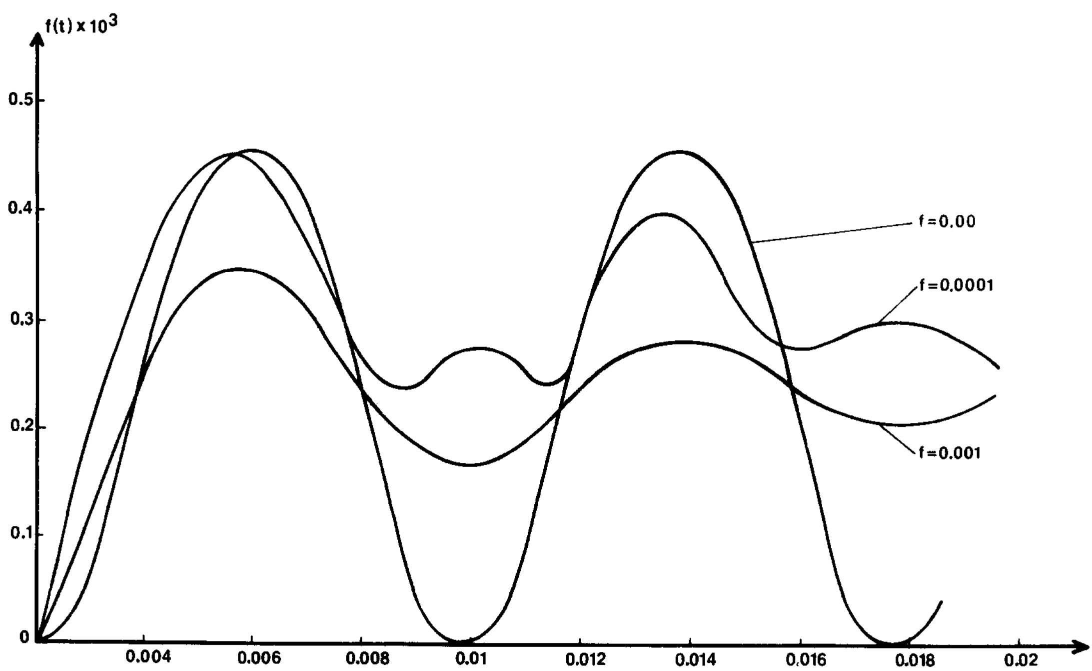

Fig. 6. Plot of the function $f(t)$ vs time for various values of the damping coefficient f.

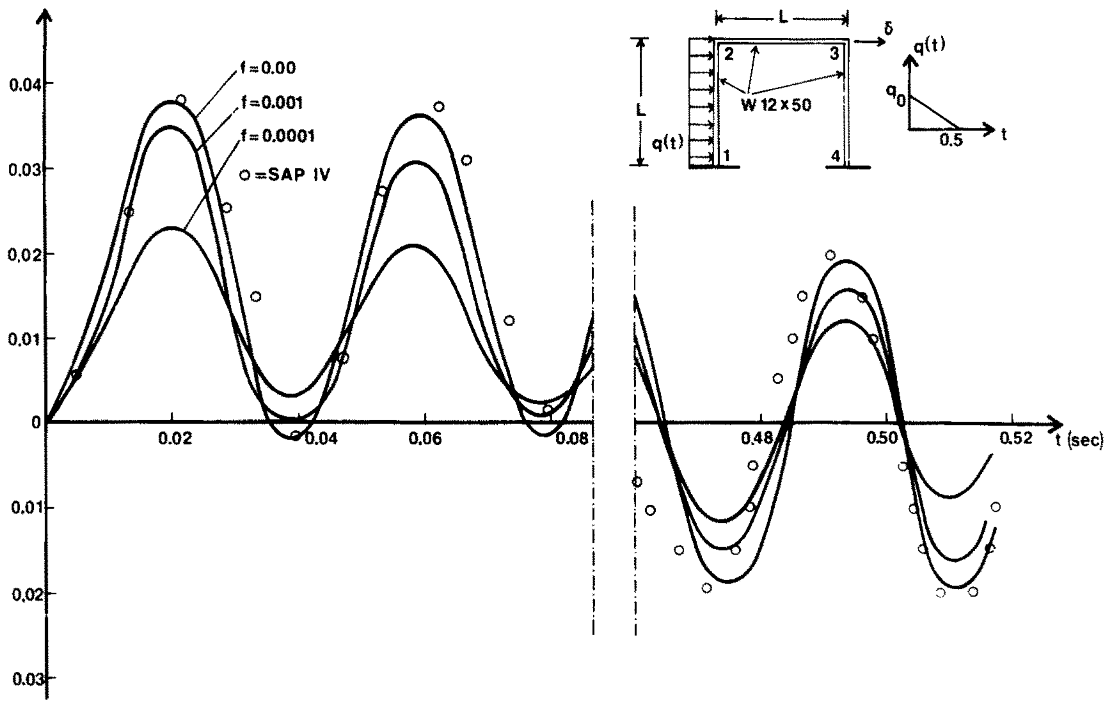

Fig. 7. Horizontal deflection $\delta_3(t)$ vs time for different amounts of damping of the portal frame of Example 2 under lateral wind forces.

total CP computer time including formulation of (44) was 2.358 sec. Figure 7(b) shows the time history of δ_2 obtained by the proposed method for various values of internal damping f. The same figure also shows results obtained by the SAP IV computer program[31] numerical integration) for four and five finite elements per member and zero damping with corresponding CP times 3.35 and 5.644. It is apparent that a four element discretization is not adequate for acceptable results. The response δ_2 obtained by the Laplace transform method in Ref. [6] coincides exactly with that obtained here for $f = 10^{-6} \approx 0$, but requires more computer time (CP time = 4.82 sec).

The dynamic response of the portal frame to wind loading was also computed by taking into account the effect of the axial deformation. In this case one has six free nodal degrees of freedom and the computation is more time consuming (CP time = 7.892 sec). The computation was done by a general computer program written in Fortran and capable of determining by the present method the dynamic response of multi-bay multi-story plane frames to dynamic nodal loading[32]. This program takes into account the effects of damping and of axial deformation. It was found that for this particular example the effect of the axial deformation is negligible as the response agrees to the previous one up to the fourth decimal figure. The CP computer time spent for the stiffness formulation, solution of the linear system in the frequency domain and inversion of δ_2 in the time domain was 7.24 sec.

Example 3

Consider a steel portal frame under a static distributed load on its girder subjected to an ideal earthquake accelogram as shown in Fig. 8(a). The horizontal deflection $\delta_2(t)$ was computed by taking into account the effect of axial deformation for different amounts of damping ($f = 10^{-6}$, 10^{-4} and 10^{-3}) and the results are shown in Fig. 8(b). For a time interval $T_0 = 5.6$ sec and $N = 1024$, the CP computer time spent for the stiffness formulation, solution of the linear system in the frequency domain and inversion of δ_2 in the time domain was 13.328 sec. The Fourier transform of the ground acceleration $\ddot{x}_s$ needed in the computation of the equivalent inertia nodal forces of (36) and (37) was determined by both the direct FFT and an exact integration of the piecewise linear function $\ddot{x}_s(t)$ and subsequent discrete evaluation. It was found that the FFT provides very good results, especially for the real part of the function $F[\ddot{x}_s]$. The response of the frame for $f = 0$ was also determined by modeling the frame as a single story shear building. This modeling of frames with very stiff girders as compared to their columns, such as the present frame, leads to a response which is very close to the exact one. Indeed it was found here that the shear building model gave results in agreement with the present exact ones to within plotting accuracy, thereby providing an approximate way to check the accuracy of the present method.

9. CONCLUSIONS

On the basis of the previous discussion the following conditions can be drawn:

(1) A general numerical method for determining the dynamic response of plane frameworks to dynamic shocks, wind forces or earthquake excitations is presented. The method formulates the dynamic problem in the frequency domain in a static-like form by using Fourier transformed dynamic stiffness coefficients $\bar{D}ij$ and solves

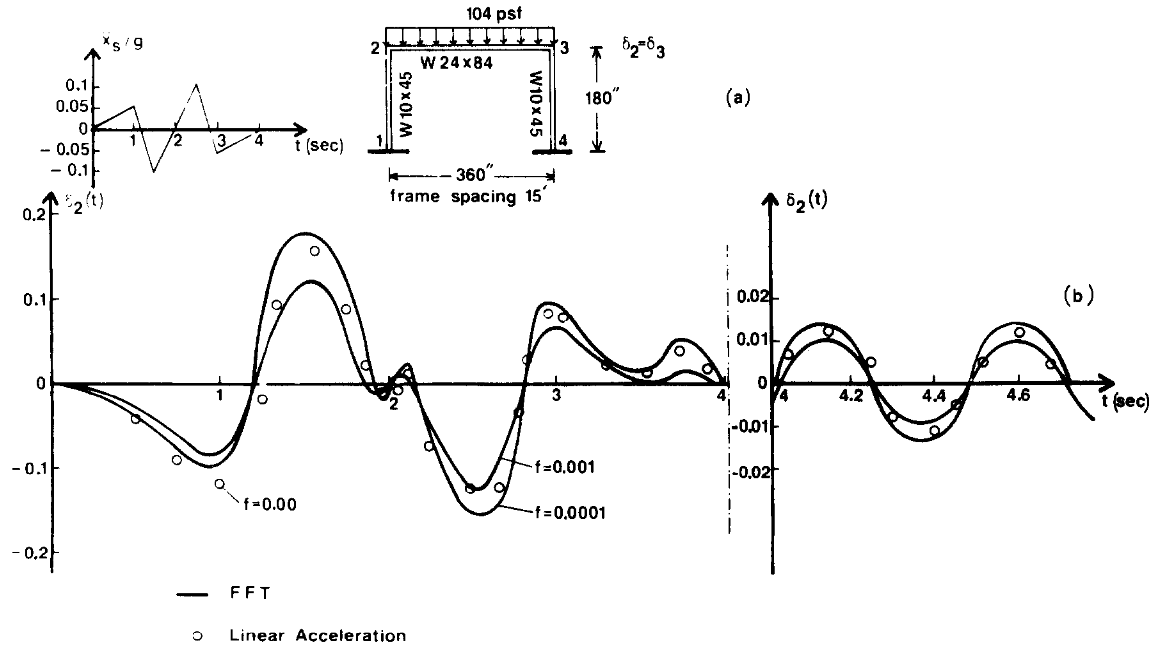

Fig. 8. Horizontal deflection $\delta_2(t)$ vs time for different amounts of damping of the portal frame of Example 2 subjected to an ideal earthquake accelogram.

it there numerically. The dynamic response is then obtained by a numerical inversion of the transformed solution.

(2) In the framework of the linear theories employed in this work the use of the $\bar{D}ij$ coefficients leads to the "exact" solution of the dynamic problem since the displacement function used for their construction is the exact solution of the transformed equations of motion. Thus, this method can provide a basis for comparing the accuracy of other approximate methods such as the conventional finite element method.

(3) Flexural and axial motion of the framework members are considered and taken into account by constructing the appropriate $\bar{D}ij$ coefficients. The effects of damping (external viscous or internal viscoelastic), axial forces on bending, rotatory inertia and shear deformation on the dynamic response are also taken into account by incorporating them into appropriately constructed $\bar{D}ij$ coefficients. It should be also noticed that the method permits consideration of different amounts of damping in different beam elements and for different motions, thus achieving a more rational representation of damping and enabling one to effectively control the response by appropriate damping changes in particular structural members.

(4) The proposed method appears to be better than the conventional finite element method in conjunction with either modal naalysis of numerical integration because it provides the exact solution of the dynamic problem of frameworks and not an approximate one as in the finite element method employing a lumped or a consistent mass representation. Besides the method does not require prior knowledge of natural frequencies and modal shapes as in modal analysis. In general, numerical integration schemes are more efficient computationally than the present method[9]. However, the present method permits one to perform numerical inversion for just one nodal displacement of interest independently of the others and in that case the amount of computer time is significantly reduced. This feature becomes more pronounced for large order systems for which the response is sought for a large interval of time. The present method based on the Fourier transform is more efficient computationally than the corresponding one based on the Laplace transform. However, the latter is better in that it can easily handle initial conditions and arbitrary amounts of damping.

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