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POWER SERIES EXPANSIONS OF DYNAMIC STIFFNESS MATRICES FOR TAPERED BARS AND SHAFTS

CONSTANTINE C. SPYRAKOS* AND CHING I CHEN†

West Virginia University, Morgantown, WV 26506-6101, U.S.A.

SUMMARY

Stiffness and consistent mass matrices for tapered bars and shafts are derived with the aid of static displacement functions. Moreover, the corresponding dynamic stiffness matrices are developed in the Laplace transform domain from the exact solutions of axial/torsional governing equations. Power series expansions of the Bessel functions comprising the dynamic stiffness influence coefficients show that the stiffness and consistent mass matrices can be mathematically derived from the dynamic stiffness matrices. A discussion on the convergence of the power series expansions is also presented. The developments provide further insight into the approximations present in conventional consistent mass formulations of frameworks with tapered members.

INTRODUCTION

The dynamic analysis of tapered beams, bars and shafts has been a subject of continuous attention and research interest. A thorough presentation of analysis methods pertinent to the dynamic behaviour of tapered beams/bars has been presented by Kolousek.[1] The conventional finite element method for determining the dynamic response of structural systems comprised of tapered flexural/axial members has been based on either a lumped or a consistent mass representation that employs as displacement functions the solutions of static governing equations.[2,3] A stepped representation of the tapered members as an assembly of uniform elements is usually adopted as the structural model. This stepped representation requires a relatively large number of elements to accurately determine the dynamic response.[4,5]

An alternative approach would be the use of an exact stiffness matrix; namely, the dynamic stiffness matrix, developed from the corresponding governing differential equations for free flexural/axial vibrations.[3,6–10] Banerjee and Williams[11] have developed exact dynamic stiffness matrices for the axial, torsional and flexural vibration of tapered beams to harmonically varying forces. However, their approach requires prior knowledge of natural frequencies and modal shapes in order to evaluate the system response to transient loads. Hallauer and Liu[12] have developed the exact dynamic stiffness matrix for a straight and uniform beam subjected to bending and torsion. The derivation of stiffness matrices for a uniform open thin-walled elastic beam under harmonic excitation has been presented by Friberg.[13] In the developments presented in References 12 and 13, the governing equations of the beam elements are solved by the method

* Associate Professor, Civil Engineering Department
† Graduate Student, Mechanical and Aerospace Engineering

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of separation of variables, and can not be extended to obtain closed form expressions of the dynamic influence coefficients for tapered beams without the use of series expansions.[14]

A highly accurate and efficient FEM formulation, based on transformed dynamic stiffness matrices, has been successfully employed by Spyrakos and Beskos[15] and Tamma *et al.*[16] and Spyrakos[17] for the dynamic analysis of frameworks modelled with uniform or tapered elements and subjected to general transient forces. In these analyses, the transformed dynamic stiffness matrices were developed in either the Fourier or the Laplace transform domain. Formulation of the transformed stiffness equation in the frequency domain leads to the evaluation of the system response from a static-like problem.[17,18] The system response can be consequently obtained in the time domain through numerical inversion.[18] Such an approach retains all the advantages of the dynamic stiffness method without requiring knowledge of natural frequencies and modal shapes. Questions concerning the relationship between the stiffness matrix for uniform beam elements developed from either static displacement functions or exact dynamic governing equations have been addressed by Paz.[19,20] He has shown that the stiffness, mass and geometric matrices used in conventional finite element treatments of dynamic and stability problems for frame structures can be derived from the dynamic stiffness matrix for a beam element.

In this study, the relationship between the stiffness and mass matrices of tapered bar/shaft elements developed from either static displacement functions, or the exact equations for free vibration axial/torsional response, is presented. It is shown that the stiffness and mass matrices of a conventional consistent mass finite element method formulation are the first two terms of a power series expansion of the corresponding dynamic stiffness matrices. The developments also allow the identification of the rate of convergence of the series expansion, thus providing the means to assess the approximations embodied in finite element formulations based on static displacement functions.

TAPERED BAR/SHAFT ELEMENT

Consider the linear tapered bar/shaft element a–b shown in Figure 1 with a straight centroidal axis and the direction of the principal axis being the same for all cross sections. The general expression describing the variation of the cross-sectional area $A(x)$ and the polar second moment of area $J(x)$ along the length is given by[3]

$$A(x) = A_a\left(1 + r\frac{x}{L}\right)^m \tag{1}$$

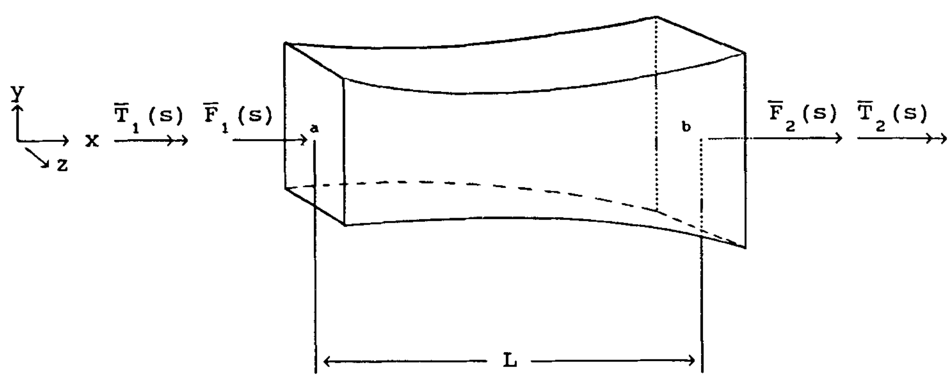

Figure 1. Geometry and end forces of tapered element

and

$$J(x) = J_a\left(1 + r\frac{x}{L}\right)^{m+2} \tag{2}$$

where

$$r = d_b/d_a - 1 \tag{3}$$

and d_i, A_i, J_i (i = a, b) denote the depth, cross-sectional area and polar second moment of area at the ends a and b of the element, respectively, L represents the element length, and m is a positive shape factor constant.

If the geometrical properties of the element at both ends of the section are given, the shape factor m can be derived from the expression

$$m = \frac{\log(A_b/A_a)}{\log(d_b/d_a)} \tag{4}$$

A rather extensive description of various cross-sectional shapes and the corresponding shape factors can be found in Reference 3. Although the developments presented in this work are valid for the whole range of variation of m, the emphasis will be placed on the practical cases of rectangular section with constant width and linear varying depth and circular sections for which $m = 1$ and $m = 2$, respectively.

STATIC STIFFNESS AND CONSISTENT MASS FORMULATION

In order to establish the relationship between a conventional finite element consistent mass formulation and a formulation based on a dynamic stiffness matrix, the derivation of the stiffness and mass matrices of a tapered member is outlined first for static displacement functions. The axial displacement, $u(x)$, and a constant axial load, P, for a tapered bar are related through

$$\frac{du}{dx} = \frac{P}{EA(x)} \tag{5}$$

where E is the Young's modulus of the bar.

Substitution of the expression for the cross-sectional area in equation (5) and subsequent integration with respect to x yields

$$u(x) = \int_0^L \frac{P}{EA_a(1 + rx/L)^m} dx + C \tag{6}$$

where C is a constant of integration.

In the following, the formulation will address the commonly encountered cases of $m = 1$ and $m = 2$ which allow an algebraically manageable treatment of the involved computations, and demonstrate the relationship between consistent mass and dynamic stiffness formulations.

The case of a tapered bar with a rectangular cross-sectional area of constant width ($m = 1$) is considered first. After performing the integration indicated in equation (6), imposing the boundary conditions, $u(x = 0) = 1$ and $u(x = L) = 0$, and introducing the variable $\xi = 1 + rx/L$, where $1 \leqslant \xi \leqslant 1 + r$, the displacement function, $u_1(x)$, takes the form

$$u_1(\xi) = 1 - \frac{\ln \xi}{\ln(1 + r)} \tag{7}$$

Analogously, the displacement function, $u_2(x)$, corresponding to $u(x = 0) = 0$ and $u(x = 1) = 1$ is given by

$$u_2(\xi) = \frac{\ln \xi}{\ln(1+r)} \tag{8}$$

Following standard finite element procedures (see e.g. Reference 2), the stiffness and consistent mass matrices can be determined from

$$k_{ij}(\xi) = \int_1^{1+r} A(\xi) E u_i'(\xi) u_j'(\xi)\, \mathrm{d}\xi \tag{9}$$

and

$$m_{ij}(\xi) = \int_1^{1+r} \bar{m}(\xi) u_i(\xi) u_j(\xi)\, \mathrm{d}\xi \tag{10}$$

where the primes indicate differentiation with respect to ξ and $\bar{m}(\xi)$ is the mass per unit length of the element.

In view of the expressions (1), (7) and (8), equations (9) and (10) yield the following stiffness and mass coefficients:

$$\begin{aligned} k_{11} &= EA_a \frac{r}{L \ln(1+r)} \\ k_{12} = k_{21} &= -EA_a \frac{r}{L \ln(1+r)} \\ k_{22} &= EA_a \frac{r}{L \ln(1+r)} \end{aligned} \tag{11}$$

and

$$\begin{aligned} m_{11} &= A_a \frac{\rho L}{r}\left[\frac{-2\ln^2(1+r) - 2\ln(1+r) + r^2 + 2r}{4\ln^2(1+r)}\right] \\ m_{12} = m_{21} &= A_a \frac{\rho L}{r}\left[\frac{(r^2+2r+2)\ln(1+r) - r^2 - 2r}{4\ln^2(1+r)}\right] \\ m_{22} &= A_a \frac{\rho L}{r}\left[\frac{2(1+r)^2(\ln^2(1+r) - \ln(1+r) + r^2 + 2r}{4\ln^2(1+r)}\right] \end{aligned} \tag{12}$$

The corresponding counterpart expressions for the displacement functions $u_1(\xi)$ and $u_2(\xi)$ along a tapered bar with a circular cross-sectional area ($m = 2$) are given by

$$u_1(\xi) = \frac{1}{r}\left(\frac{1+r}{\xi} - 1\right), \quad u_2(\xi) = \frac{1+r}{r}\left(1 - \frac{1}{\xi}\right) \tag{13}$$

Evaluation of the stiffness and mass matrices coefficients through equations (1), (9), (10) and (13) results in

$$\begin{aligned} k_{11} &= EA_a\left(\frac{1+r}{L}\right) \\ k_{12} = k_{21} &= -EA_a\left(\frac{1+r}{L}\right) \\ k_{22} &= EA_a\left(\frac{1+r}{L}\right) \end{aligned} \tag{14}$$

and

$$m_{11} = A_a \frac{\rho L}{3}$$

$$m_{12} = m_{21} = A_a \frac{\rho L(1 + r)}{6} \tag{15}$$

$$m_{22} = A_a \frac{\rho L(1 + r)^2}{3}$$

For an elastic tapered shaft with either a circular or a rectangular cross-section area, the similarity of the governing equations describing the axial and torsional deformations[21] allows the evaluation of the torsional stiffness and mass matrices from equations (11), (12), (14) and (15) by simply replacing the variables A_a, E with J_a and G, respectively, where G is the shear modulus of elasticity.

Consequently, the equation of motion of a structure comprising a tapered bar or shaft and subjected to a exciting force $\{f(t)\}$ can be expressed as

$$[M]\{\ddot{u}\} + [K]\{u\} = \{f(t)\} \tag{16}$$

where $[M]$ and $[K]$ are the consistent mass and stiffness matrices, respectively.

DYNAMIC STIFFNESS FORMULATION

The governing equation for the longitudinal motion of a tapered rod is given by

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right] - \rho A(x)\frac{\partial^2 u}{\partial t^2} = 0 \tag{17}$$

where ρ is the mass density of the bar. By introducing the parameter ξ defined in the previous section and by assuming the zero initial conditions, the application of Laplace transform with respect to time on equation (17) results in

$$\xi^2 \bar{u}'' + m\xi \bar{u}' - s^2 \frac{\rho L}{Er^2} \xi^2 \bar{u} = 0 \tag{18}$$

The general solution of equation (18) contains Bessel functions of the first and second kind with complex kernels. Following the procedure indicated by Spyrakos,[17] one can arrive at the following concise form for the general solution of equation (18):

$$\bar{u}(s) = \xi^k \left\{ C_1 I_k\left[\frac{sL}{r}\left(\frac{\rho}{E}\right)^{1/2} \xi\right] + C_2 K_k\left[\frac{sL}{r}\left(\frac{\rho}{E}\right)^{1/2} \xi\right]\right\} \tag{19}$$

where C_1 and C_2 are constants, and $k = (1 - m)/2$ denotes the order of the modified Bessel functions I_k and K_k.

The dynamic stiffness influence coefficients $\bar{k}'_{ij}$ can be obtained by imposing the boundary conditions on the element with the positive convention shown in Figure 1. In the Laplace transform domain, the stiffness equation corresponding to the axial equation of motion, equation (16), is given by[17]

$$\begin{Bmatrix} \bar{F}_1(s) \\ \bar{F}_2(s) \end{Bmatrix} = \begin{Bmatrix} \bar{k}'_{11}(s) & \bar{k}'_{12}(s) \\ \bar{k}'_{21}(s) & \bar{k}'_{22}(s) \end{Bmatrix} \begin{Bmatrix} \bar{u}_1(s) \\ \bar{u}_2(s) \end{Bmatrix} \tag{20}$$

where

$$\begin{aligned}\bar{k}'_{11} &= -H\{I_k(b)K_n(a) + I_n(a)K_k(b)\}\\ \bar{k}'_{12} = \bar{k}'_{21} &= \frac{H}{a(1+r)^k}\\ \bar{k}'_{22} &= -H(1+r)^m\{I_n(b)K_k(a) + I_k(a)K_n(b)\}\end{aligned} \tag{21}$$

with n, H, a and b given by

$$\begin{aligned}n &= -\tfrac{1}{2}(1+m)\\ H &= EA_a \frac{1}{B} s(\rho/E)^{1/2}\\ a &= \frac{sL}{r}(\rho/E)^{1/2}\\ b &= (1+r)a\end{aligned} \tag{22}$$

and

$$B = I_k(a)K_k(b) - I_k(b)K_k(a) \tag{23}$$

where B is subjected to the condition

$$I_k(a)K_k(b) - I_k(b)K_k(a) \neq 0 \tag{24}$$

The dynamic stiffness influence coefficients for the torsional vibration of a shaft, $\bar{k}''_{ij}$, $i, j = 1, 2$, can be obtained from equations (21) by replacing the variables m, a, b and H with l, α, β and D, respectively, given by Spyrakos:[17]

$$\begin{aligned}l &= m + 2\\ \alpha &= \frac{sL}{r}\left(\frac{\rho}{CG}\right)^{1/2}\\ \beta &= (1+r)a\\ D &= CGJ_a s\left(\frac{\rho}{CG}\right)^{1/2} \Big/ B\end{aligned} \tag{25}$$

POWER SERIES EXPANSION OF THE DYNAMIC MATRIX

The following developments demonstrate that the stiffness and consistent mass matrices are the first and second order terms, respectively, of a power series expansion of the dynamic stiffness matrix. This is attested by expanding in power series the Bessel functions that compose the dynamic stiffness coefficients. The series expansions of the functions that appear in a repetitive fashion in the dynamic stiffness coefficients $\bar{k}'_{ij}$ and $\bar{k}''_{ij}$ are presented in Appendix I. The ensuing power series expansions include addition, subtraction, multiplication and division. As can be found in Knopp,[22] addition, subtraction and multiplication of convergent power series leads also to series which converge at least within the common interval of the algebraically manipulated series. Caution, however, is required in dividing power series, since the range of convergence can be determined through a rigorous treatment of the complex series. In general, division of two convergent power series about a point Z_0 results in a series convergent within a circle centred at Z_0 and having as a radius the closest singularity to Z_0 of the functions expanded in series in the numerator and denominator. Commencing with the rectangular cross-sectional bar element, ($m = 1$), and utilizing the series expansions given in Appendix I, the dynamic stiffness coefficients can be expressed in terms of power series of the variable, s, as

$$
\begin{aligned}
\bar{k}'_{11} &= -H\{I_0(b)K_{-1}(a) + I_{-1}(a)K_0(b)\} = EA_a s\sqrt{\frac{\rho}{E}}\{-KT_{11}B^{-1}\} \\
&= A_a\frac{rE}{L\ln(1+r)} - A_a\frac{\rho L}{r}\left[\frac{2\ln^2(1+r) + 2\ln(1+r) - r^2 - 2r}{4\ln^2(1+r)}\right]s^2 \\
&\quad + (\rho A_a)^2\frac{L^3}{A_a E}A_{11}s^4\ldots \\
\bar{k}'_{12} &= \bar{k}'_{21} = \frac{H}{a(1+r)^0} = EA_a s\sqrt{\frac{\rho}{E}}\frac{1}{a}B^{-1} \\
&= -A_a\frac{rE}{L\ln(1+r)} + A_a\frac{\rho L}{r}\left[\frac{(r^2+2r+2)\ln(1+r) - r^2 - 2r}{4\ln^2(1+r)}\right]s^2 \\
&\quad + (\rho A_a)^2\frac{L^3}{A_a E}A_{12}s^4 + \ldots \\
\bar{k}'_{22} &= -H\{I_{-1}(b)K_0(a) + I_0(a)K_{-1}(b)\} = EA_a s\sqrt{\frac{\rho}{E}}(1+r)(-KT_{22}B^{-1}) \\
&= A_a\frac{rE}{L\ln(1+r)} + A_a\frac{\rho L}{r}\left[\frac{2(1+r)^2[\ln^2(1+r) + \ln(1+r)] + r^2 + 2r}{4\ln^2(1+r)}\right]s^2 \\
&\quad + (\rho A_a)^2\frac{L^3}{A_a E}A_{22}s^4 + \ldots
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
A_{11} &= \frac{8\ln^3(1+r) + 20\ln^2(1+r) - \ln(1+r)(5r^4 + 20r^3 + 46r^2 + 52r) + 8(r^2+2r)^2}{128r^3\ln^3(1+r)} \\
A_{12} &= \frac{3r^4 + 12r^3 + 22r^2 + 20r + 10}{64r^3} - \frac{13(r^4 + 4r^3 + 6r^2 + 4r)}{128r^3\ln^2(1+r)} + \frac{r^2 + 4r + 4}{16r\ln^3(1+r)} \\
A_{22} &= \frac{3r^3 + 9r^2 + 13r + 7}{16r^3} - \frac{3r^4 + 12r^3 + 14r^2 + 4r - 3}{16r^3\ln(1+r)} \\
&\quad - \frac{5r^4 + 20r^3 + 46r^2 + 52r + 42}{128r^3\ln^2(1+r)} + \frac{8r^2 + 32r + 32}{128r\ln(1+r)}
\end{aligned}
\tag{27}
$$

The convergence of the series expansion in equations (26) is dominated by the transcendental equation (23). The variation range for which the series are convergent is specified by equation (24), and is given in Appendix II for representative values of a and r. It should be noted that, regardless the value of r, the series are convergent for $a > 0{\cdot}12$.

For a taper bar with a circular cross-sectional area, the Bessel functions can be expressed in terms of hyperbolic functions:[23]

$$
\begin{aligned}
I_{-1/2}(z) &= (2/\pi z)^{1/2}\cosh z \\
I_{-3/2}(z) &= -\sqrt{\frac{2}{\pi z}}\left(\frac{\cosh z}{z} - \sinh z\right) \\
K_{-1/2}(z) &= e^{-z}(\pi/2z)^{1/2} \\
K_{-3/2}(z) &= e^{-z}\sqrt{\frac{\pi}{2z}}\left(\frac{1}{z} + 1\right)
\end{aligned}
\tag{28}
$$

where B and H can be represented as

$$
\begin{aligned}
B &= I_{-1/2}(a)K_{-1/2}(b) - I_{-1/2}(b)K_{-1/2}(a) \\
&= -\sqrt{\frac{1}{ab}}\sinh(ar)
\end{aligned}
\tag{29}
$$

$$
H = EA_{\mathrm{a}}\frac{1}{B}s\sqrt{\frac{\rho}{E}} = -EA_{\mathrm{a}}s\sqrt{\frac{\rho}{E}}\frac{\sqrt{ab}}{\sinh(ar)}
$$

In this case, the power expansions of the hyperbolic functions are given by

$$
\begin{aligned}
\cosh ar &= 1 + \frac{(ar)^2}{2!} + \frac{(ar)^4}{4!} + \frac{(ar)^6}{6!} + \dots \\
\sinh ar &= ar + \frac{(ar)^3}{3!} + \frac{(ar)^5}{5!} + \frac{(ar)^7}{7!} + \dots
\end{aligned}
\tag{30}
$$

and

$$
(\sinh ar)^{-1} = \frac{1}{ar} - \frac{ar}{6} + \frac{7}{360}(ar)^3 + \dots
$$

In view of the expressions (28)–(30), the dynamic stiffness coefficients can be expanded in power series as

$$
\begin{aligned}
\bar{k}'_{11} &= -H\{I_{-1/2}(b)K_{-3/2}(a) + I_{-3/2}(a)K_{-1/2}(b)\} \\
&= EA_{\mathrm{a}}s\sqrt{\frac{\rho}{E}}\left\{\frac{1+r}{ar} + \frac{ar}{3} - \frac{a^3r^3}{45} - \frac{(2r-3)a^5r^5}{1080} + \dots\right\} \\
\bar{k}'_{12} = \bar{k}'_{21} &= \frac{H}{a(1+r)^{-1/2}} = -\frac{\sqrt{1+r}}{a}EA_{\mathrm{a}}s\sqrt{\frac{\rho}{E}}\frac{\sqrt{ab}}{\sinh(ar)} \\
&= -EA_{\mathrm{a}}\frac{(1+r)}{L} + A_{\mathrm{a}}\frac{\rho L}{6}(1+r)s^2 - (\rho A_{\mathrm{a}})^2\frac{7L^3}{360A_{\mathrm{a}}E}(1+r)s^4 + \dots \\
\bar{k}'_{22} &= -H(1+r)^2\{I_{-3/2}(b)K_{-1/2}(a) + I_{-1/2}(a)K_{-3/2}(b)\} \\
&= EA_{\mathrm{a}}s\sqrt{\frac{\rho}{E}}(1+r)^2\left\{\frac{1}{ar(1+r)} + \frac{ar}{3} - \frac{a^3r^3}{45} + \frac{(2r+3)a^5r^5}{2160(1+r)} + \dots\right\}
\end{aligned}
\tag{31}
$$

It should noted that, in this case, the series expansions in equations (31) are convergent in all the real field $0 \leqslant ar \leqslant \infty$.

By comparing the series expansions of the $\bar{k}'_{ij}$ given by equations (26) and (31), it is observed that the first terms of the expansion series are equal to the corresponding k_{ij} stiffness coefficients given by equations (11) and (14), while the second terms are identical to the m_{ij} consistent mass coefficients expressed by equations (12) and (15), respectively. Consequently the dynamic stiffness matrix in equation (20) can be written in the following series form:

$$[S] = \begin{Bmatrix} \bar{k}'_{11}(s) & \bar{k}'_{12}(s) \\ \bar{k}'_{21}(s) & \bar{k}'_{22}(s) \end{Bmatrix}$$

$$= [K] + [M]s^2 + [A_1]s^4 + [A_2]s^6 \ldots \tag{32}$$

where $[K]$ and $[M]$ are the stiffness and mass matrices developed with the aid of static displacement functions. The matrices $[A_1]$ and $[A_2]$ are high order terms of the power series expansion of the dynamic stiffness matrix. According to the discussion on the similarities between bar and shaft elements, a series expansion of the dynamic stiffness matrices for a tapered shaft element will result in an expression similar to equation (32).

CONCLUSIONS

For tapered bar or shaft elements, it has been demonstrated that the stiffness and consistent mass matrices based on static displacement functions can be derived from the associated dynamic stiffness matrices. This is accomplished through a power series expansion of the Bessel functions that comprise the dynamic stiffness matrices. In all cases, the range of convergence of the series has been determined, thus identifying some approximations inherent in stiffness and consistent mass formulations based on static displacement functions. It is expected that the approach can be extended and similar conclusions can be drawn for tapered flexural beam elements; nevertheless, the computational effort is anticipated to be rather cumbersome.

APPENDIX I

The power expansion of the Bessel functions appearing in the dynamic stiffness influence coefficients is given by the following expressions.

For a rectangular cross section, $m = 1$, with $k = 0$ and $n = -1$

$$\text{let} \quad 1 + r = R \quad \text{and} \quad b = Ra$$

then

$$\begin{aligned}
I_0(a) &= 1 + \frac{1}{4}a^2 + \frac{1}{64}a^4 + \frac{1}{2304}a^6 + \ldots \\
I_{-1}(a) = I_1(a) &= \frac{1}{2}a + \frac{1}{16}a^3 + \frac{1}{384}a^5 + \ldots \\
K_0(a) &= -\left(\gamma + \ln\frac{a}{2}\right)I_0(a) + \frac{1}{4}a^2 + \frac{3}{128}a^4 + \frac{11}{5616}a^6 + \ldots \\
K_{-1}(a) &= \left(\gamma + \ln\frac{a}{2}\right)I_1(a) + \frac{1}{a} - \frac{1}{2}\left(\frac{a}{2} + \frac{5}{32}a^3 + \frac{5}{576}a^5 + \ldots\right)
\end{aligned} \tag{A1}$$

where $\gamma = 0{\cdot}5772157$ is Euler's constant. Also,

$$K_0(a) = -\frac{518\ln(a/2)+299}{518} - \frac{518\ln(a/2)-219}{2072}a^2 - \frac{259\ln(a/2)-239}{16576}a^4 + \ldots$$

$$K_{-1}(a) = \frac{1}{a} + \frac{777\ln(a/2)+60}{1554}a + \frac{3108\ln(a/2)-2901}{49728}a^3 + \ldots \tag{A2}$$

In view of equations (A1) and (A2) the following expansions can be obtained through algebraic manipulations.

$$B = I_0(a)K_0(b) + I_0(b)K_0(a)$$
$$= -\ln R - \frac{(R^2+1)\ln R - R^2 + 1}{4}a^2 - \frac{(2R^4+8R^2+2)\ln R - 3R^4 + 3}{128}a^4 + \ldots \tag{A3}$$

$$B^{-1} = -\frac{1}{\ln R} + \frac{(R^2+1)\ln R - (R^2-1)}{4\ln^2 R}a^2$$
$$- \frac{2\ln^3 R(3R^4+4R^2+3) - 13(R^4-1)\ln R + 8(R^4-2R^2+1)}{128\ln^3 R}a^3 + \ldots$$

$$KT_{11} = I_0(b)K_{-1}(a) + I_{-1}(a)K_0(b)$$
$$= \frac{1}{a} + \frac{R^2 - 2\ln R - 1}{4}a + \frac{R^4 - 8R^2\ln R + 4R^2 - 4\ln R - 5}{64}a^3 + \ldots \tag{A4}$$

$$-KT_{11}B^{-1} = \frac{1}{a\ln R} - \frac{2\ln^2 R + 2\ln R - R^2 + 1}{4\ln^2 R}a$$
$$- \frac{-8\ln^3 R - 20\ln^2 R + (5R^4+16R^2-21)\ln R - 8R^4 + 16R^2 - 8}{128\ln^3 R}a^3 + \ldots$$

$$KT_{22} = I_{-1}(b)K_0(a) + I_0(a)K_{-1}(b)$$
$$= \frac{1}{ar} + \frac{2R^2\ln R - R^2 + 1}{4R}a + \frac{4R^4\ln R - 5R^4 + 8R^2\ln R + 4R^2 + 1}{64R}a^3 + \ldots \tag{A5}$$

$$-KT_{22}B^{-1} = \frac{1}{aR\ln R} + \frac{2R^2\ln^2 R - 2\ln R + R^2 - 1}{4R\ln^2 R}a$$
$$+ \left\{\left(\frac{(24R^4+32R^2)}{128R} - \frac{(28R^4-32R^2-16)}{128R\ln R} - \frac{5R^4+16R^2+21}{128R\ln^2 R}\right.\right.$$
$$\left.\left. + \frac{8R^4-16R^2+8}{128R\ln^3 R}\right)a^3\right\} + \ldots$$

APPENDIX II

Evaluation of $B = I_k(a)K_k(b) - I_k(b)K_k(a)$

a	$r = 0{\cdot}1$	$r = 0{\cdot}2$	$r = 0{\cdot}3$	$r = 0{\cdot}4$	$r = 0{\cdot}5$
0·02	3·9332	0·0000	− 0·2624	0·0000	0·0002
0·04	0·0005	3·1543	3·0743	0·0000	2·9314
0·06	0·0011	0·0000	0·0000	0·0000	0·0000
0·08	0·0000	− 0·1823	0·0000	− 0·3365	− 0·0016
0·10	0·0000	− 0·1823	− 2·4313	− 0·3366	− 2·4347
0·12	− 0·0953	− 0·1823	− 0·2624	− 0·3366	− 0·4057

a	$r = 0{\cdot}6$	$r = 0{\cdot}7$	$r = 0{\cdot}8$	$r = 0{\cdot}9$	$r = 1{\cdot}0$
0·02	− 4·0291	0·0000	− 0·5878	3·3868	0·0000
0·04	2·8670	− 0·5307	2·7519	0·0000	2·6443
0·06	2·4664	− 2·9329	− 2·9388	2·2928	− 0·6936
0·08	2·1847	− 0·5309	− 2·6570	2·0156	− 2·6602
0·10	− 2·4365	− 0·5311	− 0·5884	− 2·4429	− 2·4453
0·12	− 0·4704	− 0·5313	− 0·5887	− 0·6431	− 0·6948

a	$r = 1{\cdot}1$	$r = 1{\cdot}2$	$r = 1{\cdot}3$	$r = 1{\cdot}4$	$r = 1{\cdot}5$
0·02	0·0000	0·0005	0·0000	0·0000	3·1126
0·04	0·0000	2·5493	2·5050	0·0000	2·4220
0·06	− 2·9419	− 0·7892	0·0000	− 2·9455	− 0·9176
0·08	− 0·7429	0·0000	0·0000	− 2·6677	1·7511
0·10	− 0·7434	− 2·4504	− 2·4531	− 2·4560	− 2·4590
0·12	− 0·7441	− 0·7912	− 0·8363	− 0·8796	− 0·9213

a	$r = 1{\cdot}6$	$r = 1{\cdot}7$	$r = 1{\cdot}8$	$r = 1{\cdot}9$	$r = 2{\cdot}0$
0·02	3·0734	3·0357	0·0000	0·0000	0·0000
0·04	2·3830	2·3454	2·3093	− 3·3464	− 1·0998
0·06	− 2·9481	− 2·9495	− 1·0317	− 1·0671	− 2·9540
0·08	− 0·9582	1·6778	− 2·6765	− 2·6789	1·5782
0·10	− 0·9597	− 2·4653	1·4360	− 2·4722	− 2·4758
0·12	− 0·9615	− 1·0003	− 1·0378	− 1·0741	− 1·1094

a	$r = 2{\cdot}1$	$r = 2{\cdot}2$	$r = 2{\cdot}3$	$r = 2{\cdot}4$	$r = 2{\cdot}5$
0·02	2·8978	0·0000	2·8382	− 4·0285	− 1·2532
0·04	− 3·3480	− 3·3489	− 1·1957	− 1·2257	− 1·2549
0·06	− 2·9556	− 2·9573	− 2·9590	0·0000	− 0·0009
0·08	0·0000	− 0·0016	0·0000	− 2·6924	1·4360
0·10	− 2·4795	− 2·4834	− 1·2047	− 2·4915	− 2·4957
0·12	− 1·1437	− 1·1770	− 1·2095	− 1·2412	− 1·2721

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