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CLOSED FORM SERIES SOLUTIONS OF BOUNDARY VALUE PROBLEMS WITH VARIABLE PROPERTIES

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Abstract—A Fourier cum polynomial series solution scheme is presented herein which is applicable to a broad class of initial/boundary value problems in engineering mechanics. Special emphasis is given to those problems that are represented by linear partial or ordinary differential equations with variable coefficients and not amenable to exact analytic solutions. In the proposed approach, the coefficients of the polynomials are simultaneously solved as functions of the undetermined Fourier coefficients by satisfying the initial/boundary conditions. A Fourier series expansion of the variable coefficients and application of orthogonality conditions leads to the evaluation of the undetermined Fourier coefficients through the solution of simultaneous equations or summation equations. The summation equations are solved in closed form by a new and highly efficient algorithm developed by the authors. Representative engineering mechanics problems are examplified to elucidate the features of the method and demonstrate its advantages over other techniques.

INTRODUCTION

Solutions of initial/boundary value problems governed by linear differential operators with variable coefficients cannot, in general, be derived in closed form. Furthermore, integral transforms can be applied selectively and the conventional techniques, if applicable, would lead to cumbersome solution algorithms[1]. Numerical methods of analysis circumvent the intractability of obtaining closed form solutions and lead to approximate solutions[2, 3]. The best known classical numerical techniques are the method of weighted residuals and variational methods[4, 5]. Application of a variational approach requires the existence of an appropriate functional which may not be available for all classes of boundary value problems. However, an approximate solution can be obtained in a set of known functions with undetermined coefficients, once the functional is established. The undetermined coefficients are evaluated by satisfying some of the boundary conditions and minimizing the functional[6]. In the method of weighted residuals (MWR), one operates directly on the differential equation and the boundary conditions. Trial functions are chosen with arbitrary coefficients which are specified by minimizing the residual. An extensive review of the research pertinent to the MWR can be found in references by Crandall[4] and Finlayson[7], while comparative studies between weighted residual methods have been presented by Fuller, Meneley, and Hetrick[8], Pomraning[9], and Shuleshko[10].

The well-known finite element, finite difference, or boundary element methods achieve similar objectives by reducing the differential equation to a system of algebraic equations[11–15]. The finite element method allows consideration of domains composed of zones of different materials each with complex properties and irregular boundaries[16]. Use of the finite difference method which appears to be as powerful as the finite element method, presents some difficulties while modelling irregular boundaries. The emerging boundary element method is, in general, more efficient than the finite element and finite difference methods, especially for problems involving infinite domains. Nevertheless, its use is limited to only problems where fundamental solutions or Green's functions are available. All these methods require a refined discretization of either the whole domain or the exterior boundary, leading to a large system of equations[15, 16].

Most of the present numerical schemes are either computationally cumbersome or restricted by their rate of convergence or complexity of boundary conditions. This is particularly true for those differential equations that have strong variations in their coefficients. However, the proposed Fourier cum polynomial series solutions can be routinely used in solving a wide range of initial boundary value problems. The method assumes a solution form which is a combination of Fourier series with undetermined coefficients and a set of polynomials with unknown constants. The assumed set of polynomials is complete and one order less than the order of the differential equation. The unknown constants are simultaneously solved as functions of the undetermined Fourier coefficients by satisfying the boundary conditions. Subsequently, the variable coefficients of the differential equation are expanded in terms of appropriate half-range Fourier series. These series of the assumed solution are inserted into the governing differential equation and the orthogonality conditions are applied. The above mathematical operations lead to an infinite summation equation as a function of the undetermined Fourier coefficients. Such an equation can be solved for the undetermined coefficients either in open form by considering the first few terms of the summation equation or through a special algorithm. In this paper, the method is presented in a generalized form which indicates a technique for solving problems with inhomogeneous boundary conditions. Examples are presented to clarify some of the concepts developed herein.

One of the distinct advantages of the proposed method over the existing weighted residual methods or numerical techniques to solve differential equations with variable coefficients is the development of a fast converging algorithm that can be truncated to obtain a desired level of accuracy, while simultaneously satisfying arbitrary boundary conditions over a finite domain. This would prevent discretization within the finite domain with variable properties. Thus, the computational efficiency of a system composed of several finite domains can be significantly increased without any loss of generalities of a given system.

BASIC METHOD

Given a system of differential equations, boundary conditions and initial conditions, the general approach is to assume a trial solution as a combination of Fourier series with undetermined coefficients and a complete set of polynomial series dependent on boundary conditions. The undetermined coefficients are found by requiring that the trial solution satisfies the differential equation, leading to an infinite set of simultaneous equations with the undetermined coefficients as unknowns. Even though, the proposed method can be employed to solve initial boundary value problems with equal ease, the treatment of a steady-state one-dimensional problem is presented herein to facilitate the understanding and highlight the most important features of the method. This is followed by the development of solutions of more complex two-dimensional problems.

Consider the following time independent equation defined in a domain Ω:

$$Lw = f, \qquad (1)$$

where L denotes a general linear differential operator involving spatial derivatives of an unknown function w, and f pertains to a given function. On the boundary S of Ω, w will have to satisty certain boundary conditions. For eqn (1), assume a trial solution of the form

$$w = \sum_{k=1} a_k \sin \frac{k\pi x}{l} + \sum_{j=1}^{n} c_j x^{j-1}, \qquad (2)$$

where a_k are undetermined coefficients, $\sin k\pi x/l$ are known linearly independent mode shapes or coordinate functions of x, c_j are constants to be determined by satisfying the boundary conditions. n is the number of boundary conditions associated with the complete set of polynomials: and l is the length of a spatial domain.

Satisfying eqn (2) with the boundary conditions results in c_j as functions of a_k and the inhomogeneous boundary quantities. This leads to

$$w = \sum_{k=1} a_k[\sin k\pi x/l + \psi_k(x)]. \qquad (3)$$

The mode shapes within the brackets of eqn (3) are a set of complete and linearly independent functions over the domain Ω and satisfy the boundary conditions.

The variable coefficients of operator L are expanded in half-range Fourier series. Substitution of the Fourier series expansion of variable coefficients and the assumed solution into eqn (1) as well as application of the orthogonality conditions leads to a summation equation in terms of the undetermined coefficients a_k.

$$\sum_{k=1}^{m} B_{i,k} a_k = \phi_k \text{ with } i = 1, (1), \infty, \qquad (4)$$

where $B_{i,k}$ and ϕ_k are known functions of the Fourier expansions of variable coefficients, boundary conditions, and the function f of eqn (1).

The degree of accuracy is dependent on the number of terms that are summed in the infinite series expression (3). This is contrary to redefining the mode shapes or adding other shape functions and using refined discretization of the domain Ω, as is done in the finite element method.

Zienkiewicz[16, 17] and others[7, 19] have pointed out that a_k may be obtained by any weighted residual method that operates on an approximation of the form given in eqn (3). Similarly, variational methods have been used to find a_k depending on the class of problems under investigation[6]. However, it has been shown in Ref. [20] by the authors that a set of simultaneous equations of fast convergence can be obtained while solving a differential equation as defined in eqn (1).

EXAMPLES

To formalize some of the aforementioned thoughts, several practical examples with variable coefficients are presented in the following sections.

Consider the problem of a beam of variable cross section with simple supports on the left end and fixed on the right (Refer to Fig. 1). Under the as-

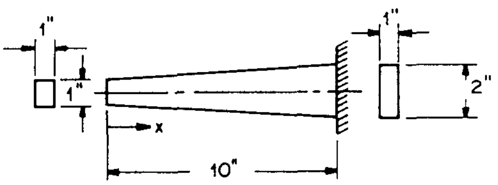

Fig. 1. Tapered beam of rectangular section.

sumptions of small deformation theory and linear elastic behavior, the flexural deflection is described by the governing equation

$$I_x \frac{d^4 y}{dx^4} + 2 I'_x \frac{d^3 y}{dx^3} + I''_x \frac{d^2 y}{dx^2} = \frac{q_x}{EI}, \quad (5a)$$

$$y(0) = y''(0) = y(l) = y'(l) = 0, \quad (5b)$$

in which I_x is the varying moment of inertia with x.

The solution form of eqn (2) is rewritten in the following manner:

$$W(x) = \sum_{k=1} a_k \sin k\pi x/l + c_1 x^3 + c_2 x^2 + c_3 x + c_4. \quad (6)$$

Invoking the boundary conditions of eqn (5b), eqn (6) can be written as

$$W(x) = \sum_{k=1} a_k [\sin k\pi x/l - k\pi/2 \cos k\pi/l (x^3/l^3 - x/l)]. \quad (7)$$

Inserting eqn (7) into eqn (5a), expanding I_x in half-range cosine series, and applying orthogonality conditions yields a summation equation similar to eqn (4) for uniform loading:

$$\sum_{k=1}^{\infty} \left[k^2 (C_{k-m} + C_{k+m} + C_0 \delta_k^m) + \frac{12}{m\pi^2} \times \left(C_m - \cos m\pi \sum_{j=1}^{\infty} \left(\frac{j^2 \cos j\pi}{4m^2 \delta_j^m + m^2 - j^2} \right) C_j \right) \right] a_k = \frac{4 w_0 l^4 (1 - \cos m\pi)}{m^3 \pi^5 E I_a}, \quad (8)$$

where

$$C_k = \frac{2}{l} \int_0^l I_x \cos k\pi x/l \, d_x = \frac{6r}{i^2 \pi^2} \times [(1 + r)^2 \cos i\pi - 1] - \frac{12 r^3}{i^4 \pi^4} (\cos i\pi - 1)$$

and

$$C_0 = \frac{1}{4r} [(1 + r)^4 - 1] \quad \text{where} \quad r = d_b/d_a - 1.$$

The above summation equation corresponding to simply supported-fixed boundary conditions is solved in open form. This is accomplished by considering several terms of the series of eqn (8). A close inspection of eqns (7) and (8) reveals that the sinusoidal part of eqn (7) and the coefficient of k^2 of eqn (8) correspond to the solution of a simply supported beam of variable thickness and the remaining terms express the effects of the fixed condition at the right support.

The center deflection of a beam with linearly varying depth of 1 to 2 in., span length of 10 in., modulus of elasticity of 30×10^3 ksi subjected to a uniform load of 0.1 k/ft is given in Table 1.

To demonstrate the applicability of the proposed method to time-dependent problems, the natural frequencies are determined for a simply supported tapered beam with depth varying from 1.5 to 2.25 in., span length of 30 in., modulus of elasticity of 30×10^6 lb/in^2, and mass density of 0.000 734 lb sec^2/in.^{4}. The natural frequencies determined from the proposed method are compared with those obtained from the exact analysis as well as those from Ref. [20].

Natural frequencies (CPS)

	Series method	Reference (20)	Exact
Mode 1	197 (First term approx.)	189.12	188.45
	192 (Two term approx.)		
Mode 2	778 (Two term approx.)	843.64	758.08

As can be noted from the above tables, only the first two or three terms of the proposed series solutions are sufficient to obtain very accurate results.

Another heuristic example that illustrates the proposed methodology herein is a time-independent two-dimensional diffusion equation with variable coefficients representing physical problems in heat conduction, soil mechanics, ground water flow, etc.

The diffusion equation for a two-dimensional domain with variable coefficients is of the form

$$\left(1 - \frac{xr}{a}\right) \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} - r/a \frac{\partial h}{\partial x} = 0 \quad (9)$$

Table 1. Center deflection in inches of a beam with linearly varying depth

No. of terms in series	With simple and fixed boundaries	With simple boundaries on both sides
Eqns (7) and (8)		
1	3.48×10^{-5}	1.24×10^{-4}
2	4.76×10^{-5}	1.403×10^{-4}
3	4.46×10^{-5}	1.399×10^{-4}
4	4.51×10^{-5}	1.400×10^{-4}
5	4.49×10^{-5}	1.402×10^{-4}
6	4.51×10^{-5}	1.402×10^{-4}
Exact	4.47×10^{-5}	1.401×10^{-4}
Finite element method 12 Elements (STRUDL)	4.46×10^{-5}	1.39×10^{-4}

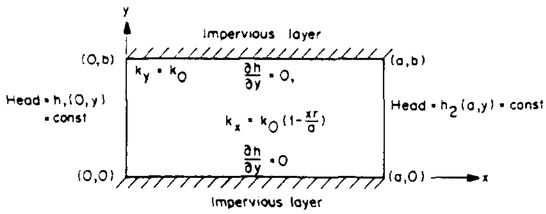

Fig. 2. Flow through anisotropic media.

with boundary conditions

$$h = h_1 \text{ and } h_2 \text{ at } x = 0 \text{ and } a, \text{ respectively}, \quad (10)$$

$$\frac{\partial h}{\partial y} = 0 \text{ at } y = 0 \text{ and } b \quad \text{(Refer to Fig. 2).}$$

The solution form of eqn (9) is written in the following manner for the above problem:

$$h(x, y) = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} h_{ij} \sin i\pi x/a + C_1 x + C_2 \right) \times \cos j\pi y/b, \quad (11)$$

where the constants C_1 and C_2 have to be obtained by satisfying the boundary conditions given in eqn (10a). It should be noted that, in this particular case, the boundary conditions (10b) are satisfied automatically because the shape function in the y direction is assumed to be of cosine series form.

By satisfying eqn (10a), C_1 and C_2 are found to be

$$C_1 = \frac{h_2 - h_1}{a} \quad \text{and} \quad C_2 = h_1.$$

Inserting eqn (11) into eqn (9) and applying orthogonality with respect to $\cos j\pi y/b$ yields

$$(1 - xr/a) \sum_i h_{ij}(i\pi/a)^2 \sin i\pi x/a + \sum_i (j\pi/b)^2 \times [h_{ij} \sin i\pi x/a + (h_2 - h_1)x/a + h_1] + (r/a) \times [\sum_i h_{ij}(i\pi/a) \cos i\pi x/a + (h_2 - h_1)/a] = 0. \quad (12)$$

Equation (12) is multiplied with $\cos k\pi x/a$ and integrated from 0 to a with respect to x to yield the following summation equation:

$$h_{ij} + \frac{2a^2}{ir} \sum_{k=1}^{\infty} k \frac{1 - \cos i\pi \cos k\pi}{k^2 - i^2 + \delta_i^k} (j/b)^2 - \frac{(1 - r)\cos i\pi \cos k\pi - 1}{k^2 - i^2 + 4i^2\delta_i^k} \left(\frac{k}{a}\right)^2 h_{kj} = \frac{2(h_2 - h_1)}{ir\pi^2} \left(\frac{ja}{ib}\right)^2 (1 - \cos i\pi). \quad (13)$$

Equation (13) is solved in closed form series for h_{ij} through a new solution algorithm that is developed by the authors and presented in the Appendix. It can be solved also in open form (Gaussian elimination, for example) by varying the dummy index i, which requires n^3 number of multiplication where "n" is the number of unknowns. However, the former approach yields an efficient algorithm for summation equations of the above nature because the number of multiplications involved are only of the order n^2.

This algorithm has salient advantages particularly for large order systems of linear equations. Values of the unknowns h_{ij} corresponding to higher magnitudes of indices can be obtained by evaluating the solution form of only a first few terms of the series solution. This is given in eqn (A2). However, a classical solution algorithm of linear equations requires a larger number of simultaneous equations to be solved to determine the values corresponding to higher order indices.

CONCLUSIONS

Closed form Fourier series solutions for boundary value problems with variable coefficients and arbitrary boundaries are developed herein. More specifically, the steady state and free vibration behavior of beams with variable properties are investigated and a set of numerical results presented. The validity of the method and the accuracy of these results are demonstrated through comparisons with the existing solutions. In addition, applicability of the proposed method to two-dimensional problems with arbitrary boundaries and varying properties is illustrated by solving a potential flow problem.

The proposed method leads to a summation equation of fast convergence, which can be solved either in open-form or in closed-form series. The closed-form series solution is developed by the authors in the form of a new and efficient algorithm[21]. The main advantage of the methodology is the development of a closed-form solution through an algorithm. This requires a lesser computational effort and leads to greater accuracy when compared to conventional methods of inversion.

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APPENDIX

Solution algorithm of summation equations:

$$h_i + \sum_{k=1}^{\bar{N}} A_{ik} h_k = \phi_i, \tag{A1}$$

where h_i is the unknown vector, A_{ik} and ϕ_i are converging (given) tensor and vector, respectively, and $\bar{N}$ is the number of unknown parameters.

The solution of h_i is

$$h_{i_0} = \phi_{i_0} + \sum_{N=1}^{\bar{N}} (-1)^N \left(\prod_{n=1}^{N} \sum_{i_n} A_{i_n, i_{n-1}} \right) \phi_{i_n}, \tag{A2}$$

where $\sum$ and $\prod$ refer to summation and product symbols and i_0 is a dummy variable taking values from 1 to N.

The detailed derivation of eqn (A2) is not presented herein due to length limitations and is given by the authors in Ref. [21].