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TIME DOMAIN BOUNDARY ELEMENT METHOD APPROACHES IN ELASTODYNAMICS: A COMPARATIVE STUDY

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Abstract—Boundary Element Method formulations of elastodynamic problems under plane strain/plane stress conditions are presented. The formulations are performed in the time domain allowing consideration of loads with transient time variation. The advantages, efficiency and accuracy of the methods are depicted through comparative studies of a representative soil-structure interaction problem.

INTRODUCTION

In recent years the Boundary Element Method (BEM) has become increasingly popular for the solution of linear elastodynamic problems [1]. Its popularity can be attributed primarily to the reduction of dimensionality of the problems, high accuracy of results and automatic consideration of the radiation conditions at infinity [2].

The earlier BEM studies of wave propagation problems have been based on frequency domain formulations. Representative are the works of Banaugh and Goldsmith [3] and Niwa *et al.* [4] in steady-state wave propagation problems. Cruse and Rizzo [5] and Manolis and Beskos [6] have successfully applied the BEM in conjunction with the Laplace transform to study plane wave propagation and stress concentration problems.

Integral equation solutions of elastodynamic problems in the time domain have been presented by Cole *et al.* [7] and Mansur and Brebbia [8] for the scalar wave equation. Niwa *et al.* [9] and Manolis [10] employed a three-dimensional BEM to investigate the scattering of elastic waves around cylindrical openings under conditions of plane strain. Spyrakos and Beskos [11–13] developed a highly efficient BEM formulation in conjunction with the two-dimensional Stokes' solution for the infinite space to study the transient response of surface and embedded strip-foundations subjected to external forces and/or obliquely incident seismic waves. A two-dimensional BEM based on weighted residual considerations to solve wave propagation problems under zero initial conditions and zero body forces has been recently presented by Mansur [14]. Commencing from similar considerations, Antes [15] derived a BEM approach to treat general two-dimensional elastodynamic problems including the contribution of initial conditions and body forces.

The present study focuses on a close examination of the available two-dimensional time domain BEM approaches dealing with general plane stress/plane strain elastodynamic problems. Besides a presentation of the integral formulations, comparative studies are presented based on the solution of a representative soil-structure interaction problem. This is followed by a discussion of the obtained results. Considerable effort is directed to ensure a comparison as fair as possible.

BASIC THEORY

Under the assumption of small displacement theory and conditions of plane strain, the motion in a homogeneous, isotropic, linear elastic medium is governed by the Navier–Cauchy equations

$$(c_1^2 - c_2^2)u_{i,ij} + c_2^2 u_{j,ii} - \ddot{u}_j = -\frac{1}{\rho} b_j, \quad i,j = 1,2, \quad (1)$$

where dots and commas indicate time and space differentiations, respectively, b_j is the body force, ρ denotes the mass density of the medium; c_1 and c_2 are the dilatational and shear wave velocities, respectively, which for the case of plane strain are given in terms of the Lamé constants λ and μ by

$$c_1 = \sqrt{[(\lambda + 2\mu)/\rho} \quad \text{and} \quad c_2 = \sqrt{(\mu/\rho)}. \quad (2)$$

For the case of plane stress the λ has to be replaced by $\lambda' = 2\lambda\mu/(\lambda + 2\mu)$. In a well-posed boundary value problem eqn (1) must be accompanied by appropriate initial and boundary conditions:

$$u_i(\mathbf{x},t)=\bar{u}_{i0}(\mathbf{x});$$

$$\dot{u}_i(\mathbf{x},t)=\bar{v}_{i0}(\mathbf{x}) \quad \text{for} \quad t=t_0 \quad \text{in} \quad \Omega+\Gamma$$

$$u_i(\mathbf{x},t)=\bar{u}_i(\mathbf{x},t) \quad \text{for} \quad t>t_0 \quad \text{on} \quad \Gamma_1$$

$$t_i(\mathbf{x},t)=[\delta_{ik}(c_1^2-2c_2^2)u_{j,j}+c_2^2(u_{i,k}+u_{k,i})]n_k$$

$$=\bar{T}_i(\mathbf{x},t) \quad \text{for} \quad t>t_0 \quad \text{on} \quad \Gamma_2, \tag{3}$$

where $\Gamma=\Gamma_1+\Gamma_2$ and Ω denote the boundary and the interior of the domain, respectively, and the bar indicates that the values are known.

The fundamental solution of eqn (1), the response of an infinite medium to a unit impulse uniformly distributed along the line perpendicular to the plane Ω at a point ξ and acting at time τ, is given by [15, 16]

$$U_j^{(i)}(\mathbf{x},\xi;t')=\frac{1}{2\pi\rho}\left\{\frac{1}{c_1}\frac{H(c_1t'-r)}{2}\left[\frac{2c_1^2t'^2-r^2}{R_1}\times r_{,i}r_{,j}-R_1\delta_{ij}\right]-\frac{1}{c_2}\frac{H(c_2t'-r)}{r^2}\times\left[\frac{2c_2^2t'^2-r^2}{R_2}r_{,i}r_{,j}-\left(R_2+\frac{r^2}{R_2}\right)\delta_{ij}\right]\right\}$$

$$=H(c_1t'-r)\,g_j^{1(i)}+H(c_2t'-r)\,g_j^{2(i)}$$

$$=U_j^{1(i)}+U_j^{2(i)}, \tag{4}$$

$$R_\alpha=(c_\alpha^2t'^2-r^2)^{1/2}, \quad \alpha=1,2$$

$$t'=t-\tau, \tag{5}$$

where H is the Heaviside function and r is the distance $|\mathbf{x}-\xi|$ between a field point $\mathbf{x}$ and a source point ξ. The fundamental solution, eqn (4), can be incorporated in integral formulations of boundary value problems in the time domain and under conditions of plane strain. This subject is elaborated in the following sections.

INTEGRAL FORMULATION I

Combining the fundamental solution, eqn (4), with the elastodynamic reciprocal theorem, eqn (1) with the conditions (3) can be replaced by the following integral identity [16]:

$$c_{ij}u_j(\xi,t)=\oint_\Gamma(U_j^{(i)}*t_{(n)j}-T_j^{(i)}*u_j)\,\mathrm{d}\Gamma+\int_\Omega(U_j^{(i)}*b_j)\,\mathrm{d}\Omega+\rho\int_\Omega<\bar{v}_{j0}(U_j^{(i)})_0-\bar{u}_{j0}(\dot{U}_j^{(i)})_0>\mathrm{d}\Omega, \tag{6}$$

where on a smooth boundary ($\xi\in\Gamma$) $c_{ij}=\delta_{ij}/2$ and for interior points ($\xi\in\Omega$) $c_{ij}=\delta_{ij}$; the operation $*$ denotes time convolution and the tensor $T_j^{(i)}$, resulting from $U_j^{(i)}$ through appropriate spatial differentiations, is given in [16]. In order to simplify the presentation of the numerical treatment of eqn (6), it is assumed that the body forces as well as the initial displacements and velocities are zero. Thus, the last two integrals in eqn (6) vanish.

The proposed solution process consists of the following two steps [11, 13]:

(i) evaluation of the response u_j due to a single rectangular impulse traction $t_{(n)i}$, and

(ii) superposition of the individual impulse responses to obtain the total system response.

These steps are accomplished through a spatial discretization requiring division of the boundary Γ into Q line elements, and a time discretization approximating the time variation of the displacements (as well as the tractions) as a sequence of rectangular impulses of equal duration Δt. The spatial and time variation of displacements and tractions over each element and time-step are assumed to be constant, even though the adaption of linear or higher-order variations is possible at the expense of additional computational effort.

The rectangular impulse traction acting on an element at the time $\tau=n\,\Delta t$ can be expressed as

$$t_{(n)i}(\mathbf{x},\tau)=\{H[t'-(n-1)\Delta t]-H[t'-n\Delta t]\}\,t_i^n(\mathbf{x}), \tag{7}$$

where t_i^n denotes the intensity of the traction vector at time $\tau=(n-0.5)\Delta t$.

In view of eqn (7), eqn (4) takes the form

$$G_{ij}^{nq}=\frac{1}{2\pi\rho}\left\{\frac{r_{,i}r_{,j}}{r^4}[H(t'-r/c_1)\,S_1^{(1)}-H(t'-r/c_2)\,S_2^{(1)}]-\frac{\delta_{ij}}{r^2}[H(t'-r/c_1)\,S_1^{(2)}-H(t'-r/c_2)\,S_2^{(2)}]+\frac{\delta_{ij}}{C_2^2}H(t'-r/c_2)\,S_2^{(3)}\right\}, \tag{8}$$

with $n=1,2,\ldots,N$ and $q=1,2,\ldots,Q$. The functions $S_i^{(j)}$ $(i=1,2;\ j=1,2,3)$ are given explicitly in Appendix I. The corresponding discretized form F_{ij}^{nq} of the tensor $T_j^{(i)}$ can be obtained through spatial differentiations as indicated in [13, 16].

Finally, with the aid of eqn (7) and the use of nodal collocation to account for the spatial variation of displacement and tractions at the boundary, the total system response to a transient loading can be evaluated from the following discretized form of eqn (6):

$$c_{ij}u^{Np}=\sum_{q=1}^{Q}\sum_{n=m}^{N}\left(\left[\int_{\Delta s}G_{ij}^{nq}\mathrm{d}s\right]\{t^{N-l+1}\}-\left[\int_{\Delta s}F_{ij}^{nq}\mathrm{d}s\right]\{u^{N-l+1}\}\right), \tag{9}$$

where p denotes the center of an arbitrary surface element and $l = n - m + 1$.

The discretized forms of the kernals G_{ij}^{nq} and F_{ij}^{nq} are characterized by properties of causality, reciprocity and translation. These are appropriately used to reduce the computations involved in eqn (9). In a well-posed boundary value problem, eqn (9) can be employed together with the prescribed displacements and tractions to obtain the remaining unknown boundary quantities. Once the boundary value problem is solved, the displacement and traction fields in the interior of the domain Ω can be easily evaluated [1, 2].

INTEGRAL FORMULATION II

An alternative general form of the integral eqn (6) replacing eqns (1) and (3) can be obtained by using weighted residual considerations in space and time [2, 14, 15]

$$c_{ij}u_j(\xi,t) = \int_0^{t^+}\left[\oint_\Gamma (U_j^{(i)}t_j - T_j^{(i)}u_j)\,\mathrm{d}\Gamma + \int_\Omega U_j^{(i)}b_j\,\mathrm{d}\Omega\right]\mathrm{d}\tau + \rho\int_\Omega < \bar{v}_{j0}(U_j^{(i)})_0 - \bar{u}_{j0}(\dot{U}_j^{(i)})_0 > \mathrm{d}\Omega, \tag{10}$$

where t^+ pertains to the $t + \epsilon$ with ϵ being arbitrarily small.

The second term in the first integral of eqn (10) is unsuitable for numerical solution due to the singularities appearing in the products of Dirac's distribution $\delta(c_\alpha t' - r)$ with $g_j^{\alpha(i)}$. In order to eliminate these singularities, the term $T_j^{(i)}u_j$ is integrated by parts with respect to time [15]:

$$\begin{aligned}
\int_0^{t^+}\oint_\Gamma T_j^{(i)}u_j\,\mathrm{d}\tau
&= \int_0^{t^+}\oint_\Gamma \rho\sum_{\alpha=1}^{2} H(c_\alpha t' - r)\\
&\times\left[(c_1^2 - 2c_2^2)\times\delta_{jk}\left(\frac{\partial}{\partial x_1}g_1^{\alpha(i)} - \frac{r_{,1}}{c_\alpha}\frac{\partial}{\partial\tau}g_1^{\alpha(i)}\right)\right.\\
&+ c_2^2\left\{\frac{\partial}{\partial x_j}g_k^{\alpha(i)} - \frac{r_{,j}}{c_\alpha}\frac{\partial}{\partial\tau}g_k^{\alpha(i)}\right.\\
&\left.\left.+\frac{\partial}{\partial x_k}g_j^{\alpha(i)} - \frac{r_{,k}}{c_\alpha}\frac{\partial}{\partial\tau}g_j^{\alpha(i)}\right\}\right]n_k u_j\\
&\times\mathrm{d}\Gamma\,\mathrm{d}\tau + \oint_\Gamma\rho\sum_{\alpha=1}^{2}\frac{1}{c_\alpha}\left[(c_1^2 - 2c_2^2)\delta_{jk}r_{,l}U_{l0}^{\alpha(i)}\right.\\
&\left.+ c_2^2\left\{r_{,k}U_{j0}^{\alpha(i)} + r_{,j}U_{k0}^{\alpha(i)}\right\}\right]n_k u_{j0}\\
&\times\mathrm{d}\Gamma - \int_0^{t^+}\oint_\Gamma\rho\sum_{\alpha=1}^{2}\frac{1}{c_\alpha}\left[(c_1^2 - 2c_2^2)\delta_{jk}r_{,l}U_l^{\alpha(i)}\right.\\
&\left.+ c_2^2\left\{r_{,k}U_j^{\alpha(i)} + r_{,j}U_k^{\alpha(i)}\right\}\right]n_k\dot{u}_j\,\mathrm{d}\Gamma\,\mathrm{d}\tau.
\end{aligned} \tag{11}$$

Similar singularities appear in the part of the volume integral containing the term $(\dot{U}_j^{(i)})_0$. Using a transformation in polar co-ordinates and integrating by parts, Antes [15] derived an expression amenable to numerical solution that accounts for the effect of initial conditions and body forces.

Herein, as stated in the integral formulation I, the initial displacements and velocities as well as the body forces are considered to be zero. This assumption facilitates the demonstration of the salient features of the numerical treatment of eqn (11). In addition, the spatial and time variation of the unknown displacements and tractions are approximated by the set of interpolation functions:

$$u_j(r,\tau) = \sum_p\sum_m \phi^p(r)\eta_m(\tau)u_{jp}^m$$

$$t_j(r,\tau) = \sum_p\sum_m \psi^p(r)\mu_m(\tau)T_{jp}^m, \tag{12}$$

where p and m pertain to space and time, respectively. In order to maintain as many common features as possible with the integral formulation I, the interpolation functions $\phi^p(r)$, $\psi^p(r)$ and $\mu_m(\tau)$ are assumed to be piecewise constant and $\eta_m(\tau)$ piecewise linear. It should be noted that a piecewise linear or higher-order variation of $\eta_m(\tau)$ is necessary to sustain the unknown velocity term $\dot{u}_j$ in eqn (11). Then, according to the approximations adopted ($N := T_n$, $S := T_s$), the integral identity (10) can be written in the following convenient for numerical computations form:

$$\begin{aligned}
c_{ij}u_j(\xi,t_n) &= \sum_p\int_{\Gamma_p}\sum_m[\mathbf{U}_j^{1(i)m}(r,\xi;t_n) + \mathbf{U}_j^{2(i)m}(r,\xi;t_n)]\\
&\times[N_p^m n_j(r) + S_p^m s_j(r)]\,\mathrm{d}r\\
&-\sum_p\int_{\Gamma_p}\sum_m[\mathbf{T}_j^{1(i)m}(r,\xi;t_n) + \mathbf{T}_j^{2(i)m}(r,\xi;t_n)]\\
&\times[u_{np}^m n_j(r) + u_{sp}^m s_j(r)]\,\mathrm{d}r\\
&+\int_0^{t_n^+}\int_\Omega U_j^{(i)}b_j\,\mathrm{d}\Omega\,\mathrm{d}\tau.
\end{aligned} \tag{13}$$

The two-dimensional kernel $\mathbf{U}_j^{\alpha(i)m}(r,\xi;t_n)$, $(\alpha = 1, 2)$, is given explicitly in Appendix II, while the expressions for the kernel $\mathbf{T}_j^{\alpha(i)m}(r,\xi;t_n)$ can be found in Ref. [15].

The above equation corresponds to eqn (9) and can be incorporated in a time-stepping algorithm together with appropriate boundary and equilibrium considerations to solve general two-dimensional elastodynamic problems.

NUMERICAL EXAMPLE

The methods presented in the previous sections are applied to obtain the response of a surface rigid strip-foundation. The foundation is subjected to external forces of a transient time variation as shown in Fig. 1. This example serves as a means for a comparison study of the time domain two-dimensional BEM formulations presented. A detailed formulation of the problem requiring the combination of eqn (9) or its counterpart eqn (13) with equilibrium and compatibility considerations at the soil foundation interface, has been described in [11–13]. Thus, the following discussion emphasizes the primary differences appearing in the implementation of the methods.

Consider a massless strip-foundation of 5-ft (1.524-m) width, ($2b$), subjected to a vertical rectangular impulse force of intensity 180 k/ft (2626.92 kN/m). The supporting homogeneous, linear elastic soil medium is characterized by a modulus of elasticity $E = 2.58984 \times 10^9$ lb/ft^2 (1.24×10^{11} N/m^2), mass density $\rho = 10.368$ lb/sec^2/ft^4 (5362.45 kg/m^3) and Poisson's ratio $\nu = 1/3$. Under conditions of plane strain, the material properties of the soil are related to the Lamé constants appearing in eqn (2) through

$$\lambda = E\nu/(1+\nu)(1-2\nu), \mu = E/2(1+\nu). \quad (14)$$

In all cases the soil foundation interface is discretized into eight equal elements, as shown in Fig. 2. It should be mentioned that results of acceptable accuracy can also be obtained for a five-element discretization. Relaxed boundary conditions are assumed at the contact area. This assumption eliminates the requirement of the infinite space fundamental solution [4] for modeling the surrounding soil surface. In addition, it permits decoupling of the vertical, horizontal and rocking motions. Thus, the amount of computational effort is considerably reduced without any significant loss of solution accuracy [11–13].

In Fig. 3, the foundation vertical impulse response is plotted for the first 32×10^{-4} sec. It clearly shows

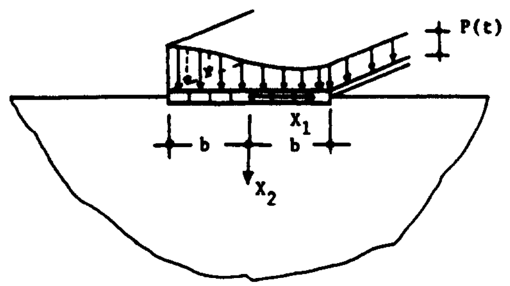

Fig. 1. Surface rigid strip-foundation to transient forces.

that the results obtained by the two methods are almost identical. Similar close agreement has been found for the horizontal and rocking motions as well [13]. In the first formulation a time-step $\Delta t = 16 \times 10^{-6}$ sec is used. During one time-step, the P-wave travels a distance equal to half the length of an element. The second formulation allows a choice of the time step with duration $\beta\Delta t$, where the factor β can be any real number ranging from 1 to 2. However, solution accuracy is assured for $\beta = 2$, which allows the second approach to determine the foundation response at time intervals twice as long as those of the first approach.

Most of the computational effort can be attributed to the integrations of the kernels in eqns (9) and (13). In all cases the integrations with respect to time were done analytically, and the spatial integrations were performed by a six point Gaussian quadrature algorithm. The computations were performed on a Cyber CDC 74 and a CDC 855 for formulations I and II, respectively. In order to increase the accuracy of formulation I, as far as the dynamic effect of the propagating waves on the response is concerned, every element is further discretized into five subelements. Such subdivision is not required in formulation II. For formulation I improvement of the solution accuracy can be achieved only through a more refined discretization of the interface or element subdivision. In contrast, formulation II provides the alternative to improve the solution accuracy by utilizing higher-order time and spatial variations

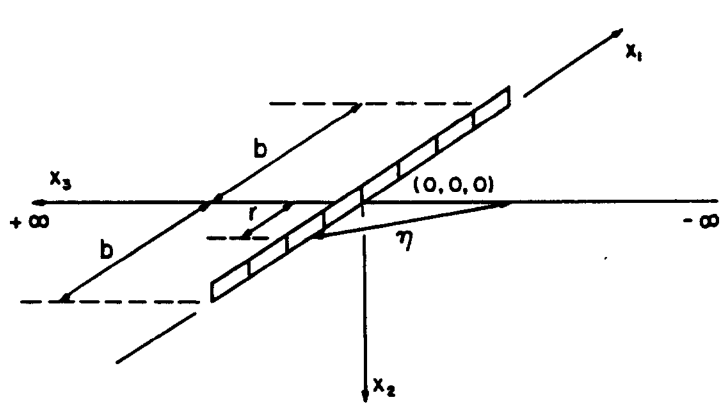

Fig. 2. Spatial discretization of a surface strip-foundation.

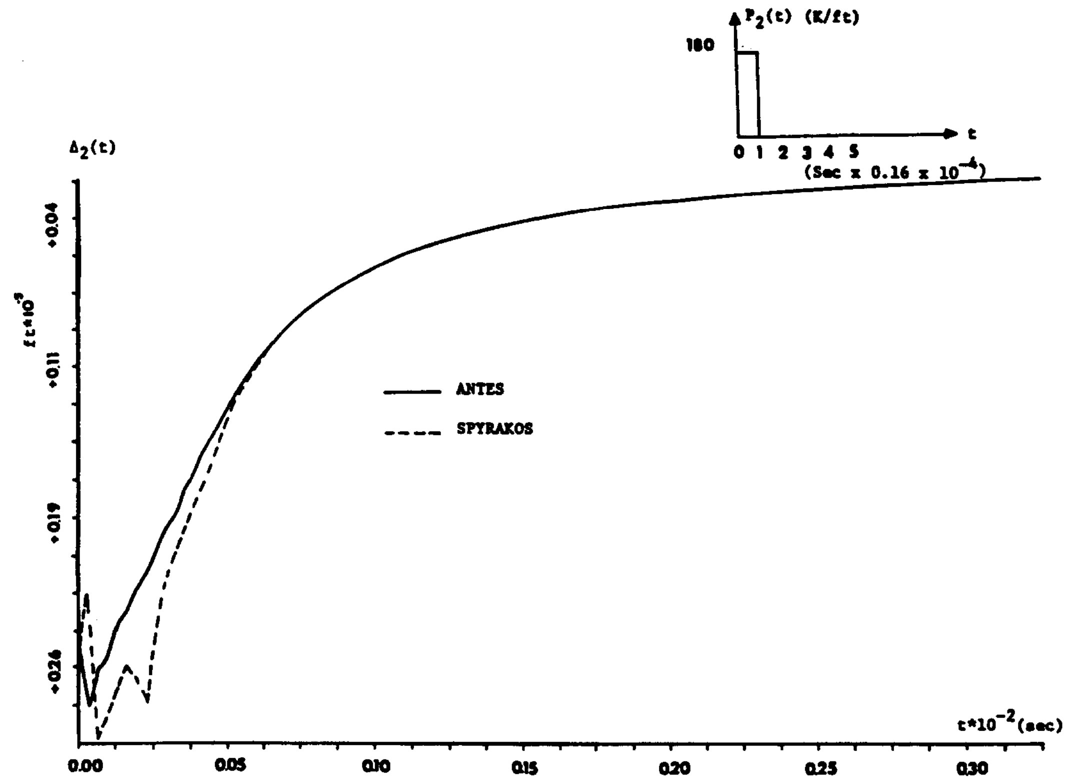

Fig. 3. Vertical response to impulse loading.

of the unknown quantities. An extension of the first approach to incorporate higher-order variations is possible but is not presented herein.

In two-dimensional BEM formulations an additional difficulty appears for r approaching zero and $c_\alpha t'$ ($\alpha = 1, 2$) being different from zero. It pertains to the singularities of the kernel functions in eqns (9) and (13). These singularities, however, are either apparent ones, which are eliminated when contributions of similar terms referring to dilatational and shear waves are evaluated together, or logarithmic that can be easily treated analytically [2]. The first BEM approach requires 150 CP sec to determine 100 time-steps of the foundation impulse vertical, horizontal and rocking motions. The evaluation of an equal number of time-steps by the second BEM approach required 69 CP sec. Provided that there is a CP ratio of 2 between the two machines, the CDC 855 being faster than the CDC 74, the two methods appear to require almost equal computational effort to evaluate the foundation impulse response. In formulation I the assumption of constant time and spatial variation of the displacements and tractions as well as the simplicity of the expressions $S_\alpha^{(j)}$ in Appendix I permit an analytical evaluation of the spatial integrations. This leads to a considerable reduction of the numerical computations [13] and, therefore, to a much smaller CP time. Similar evaluations, even though more cumbersome, are possible in formulation II.

The harmonic, vertical, horizontal and rocking motions for the foundation subjected to the external forces

$$P_1(t) = P_{10} \sin \omega t, \quad P_2(t) = P_{20} \sin \omega t,$$

$$M_3(t) = M_{30} \sin \omega t,$$

where $P_{10} = P_{20} = 180$ k/ft (2626.92 kN/m), $M_{30} = 180$ k and $\omega = 5814$ rad are plotted in Fig. 4. For the evaluation of the first 32×10^{-4} sec of the foundation response, formulation II requires 69 CP sec, while formulation I only 0.95 CP sec. This considerable difference is attributed to the algorithms used in the two methods. The first time domain BEM approach employs the foundation impulse response in conjunction with the superposition principle to obtain the response of the foundation to any external load of a transient time variation. In contrast, the second method employs the same algorithm used for the evaluation of the foundation impulse response to determine the response due to transient loads.

CONCLUSIONS

On the basis of the preceding discussion of the time domain BEM formulations and their application on a representative soil-structure interaction problem, the following conclusions can be drawn:

1. The general BEM approaches presented can be successfully employed to solve linear elastodynamic

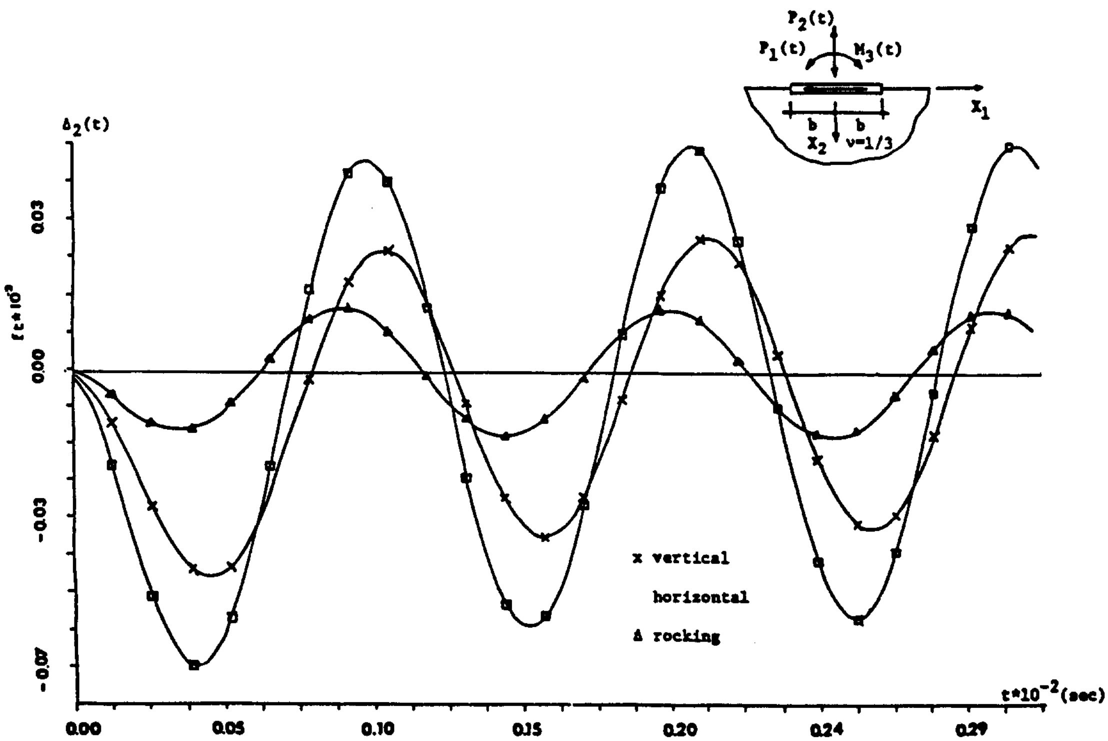

Fig. 4. Harmonic forces response.

problems under plane strain/plane stress conditions. The formulation I appears to be very efficient once the impulse system response is obtained. The formulation II permits the approximation of the unknown quantities with linear or higher-order interpolation functions.

2. In both methods improvement of the solution accuracy can be achieved with more refined time or boundary discretization. The first approach requires element subdivision to yield accurate results. Since the space and time discretizations are interrelated, special consideration should be given to the appropriate time-step.

3. The solution algorithms do not become evolved in later time-steps and are accurate for both short and long time intervals.

4. The methods provide the time response directly, not in two steps as frequency domain approaches require.

5. An advantage of time domain BEM over frequency domain approaches is that they are more suitable for extension to nonlinear material behavior.

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APPENDIX I

The kernels $S_\alpha^{(j)}$ appearing in eqn (8) are given by

1. For $0 < t - r/c_\alpha \leqq \Delta t$, $(\alpha = 1, 2)$

$$S_\alpha^{(1)} = t'\sqrt{[t'^2 - (r/c_\alpha)^2]},$$

$$S_\alpha^{(2)} = 0.5\left[t'\sqrt{[t'^2 - (r/c_\alpha)^2]} - (r/c_\alpha)^2 \times \ln\frac{t' + \sqrt{[t'^2 - (r/c_\alpha)^2]}}{r/c_\alpha}\right]$$

$$S_2^{(3)} = \ln\frac{t' + \sqrt{[t'^2 - (r/c_\alpha)^2]}}{r/c_\alpha}.$$

2. For $\Delta t < t - r/c_\alpha$

$$S_\alpha^{(1)} = t'\sqrt{[t'^2 - (r/c_\alpha)^2]} - (t' - \Delta t) \times \sqrt{[(t' - \Delta t)^2 - (r/c_\alpha)^2]},$$

$$S_2^{(3)} = \ln\frac{t' + \sqrt{[t'^2 + (r/c_\alpha)^2]}}{t' - \Delta t + \sqrt{[(t' - \Delta t)^2 + (r/c_\alpha)^2]}},$$

$$S_\alpha^{(2)} = 0.5\left\{t'\sqrt{(t'^2 - (r/c_\alpha)^2]} - (t' - \Delta t)\sqrt{[(t' - \Delta t)^2 - (r/c_\alpha)^2} - (r/c_\alpha)^2 \times \ln\frac{t' + \sqrt{[t'^2 + (r/c_\alpha)^2]}}{t' - \Delta t + \sqrt{[(t' - \Delta t)^2 + (r/c_\alpha)^2]}}\right\}.$$

APPENDIX II

The following kernel appears in eqn (13):

$$U_j^{\alpha(i)m}(r, \xi; t_n)$$

$$= \frac{1}{2\pi\rho}\Bigg(r^{-2}(\tfrac{1}{2}\delta_{ij} - r_{,i}r_{,j})(-1)^\alpha \times \{(t_n - t_m + \Delta t)\sqrt{[(t_n - t_m + \Delta t)^2 - (r/c_\alpha)^2]} - (t_n - t_m)\sqrt{[(t_n - t_m)^2 - (r/c_\alpha)^2]}\} + \frac{\delta_{ij}}{2c_\alpha^2}\ln\frac{(t_n - t_m + \Delta t) + \sqrt{[(t_n - t_m + \Delta t)^2 - (r/c_\alpha)^2]}}{t_n - t_m + \sqrt{[(t_n - t_m)^2 - (r/c_\alpha)^2]}}\Bigg)$$

if $(t_n - t_m) > r/c_\alpha$

$$= \frac{1}{2\pi\rho}\Bigg(r^{-2}(\tfrac{1}{2}\delta_{ij} - r_{,i}r_{,j})(-1)^\alpha \times \{(t_n - t_m + \Delta t)\sqrt{[(t_n - t_m + \Delta t)^2 - (r/c_\alpha)^2]}\} + \frac{\delta_{ij}}{2c_\alpha^2}\ln\frac{(t_n - t_m + \Delta t) + \sqrt{[(t_n - t_m + \Delta t)^2 - (r/c_\alpha)^2]}}{r/c_\alpha}\Bigg)$$

if $(t_n - t_m) < r/c_\alpha < t_n - t_m + \Delta t$

$= 0$ elsewhere.